

This manual includes

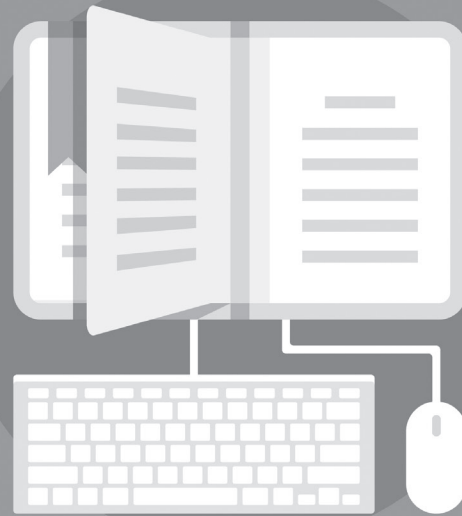
GOAL

Customizable, versatile
online exam question bank.
Thousands of questions!

Access your exclusive
StudyPlus⁺ bonus content:
GOAL | Flashcards | Formula sheet
* Key Code Inside *

a/s/m

Exam LTAM Study Manual



1st Edition, 3rd Printing

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

NO RETURN IF OPENED

a/s/m

Actuarial Study Materials
Learning Made Easier

TO OUR READERS:

Please check A.S.M.'s web site at www.studymanuals.com for errata and updates. If you have any comments or reports of errata, please e-mail us at mail@studymanuals.com.

©Copyright 2018 by Actuarial Study Materials (A.S.M.), PO Box 69,
Greenland, NH 03840. All rights reserved. Reproduction in whole or in part
without express written permission from the publisher is strictly prohibited.

Contents

1	Probability Review	1
1.1	Functions and moments	1
1.2	Probability distributions	2
1.2.1	Bernoulli distribution	3
1.2.2	Uniform distribution	3
1.2.3	Exponential distribution	3
1.3	Variance	4
1.4	Normal approximation	5
1.5	Conditional probability and expectation	6
1.6	Conditional variance	8
	Exercises	9
	Solutions	12
2	Introduction to Long Term Insurance	17
I	Survival Models	19
3	Survival Distributions: Probability Functions	21
3.1	Probability notation	21
3.2	Actuarial notation	24
3.3	Life tables	25
3.4	Number of survivors	27
	Exercises	29
	Solutions	33
4	Survival Distributions: Force of Mortality	39
	Exercises	43
	Solutions	52
5	Survival Distributions: Mortality Laws	61
5.1	Mortality laws that may be used for human mortality	61
5.1.1	Gompertz's law	64
5.1.2	Makeham's law	64
5.1.3	Weibull Distribution	65
5.2	Mortality laws for easy computation	66
5.2.1	Exponential distribution, or constant force of mortality	66
5.2.2	Uniform distribution	66
5.2.3	Beta distribution	67
5.3	British mortality tables	68
	Exercises	68
	Solutions	72
6	Survival Distributions: Moments	77
6.1	Complete	77
6.1.1	General	77
6.1.2	Special mortality laws	79
6.2	Curtate	82

Exercises	86
Solutions	94
7 Survival Distributions: Percentiles and Recursions	107
7.1 Percentiles	107
7.2 Recursive formulas for life expectancy	108
Exercises	109
Solutions	113
8 Survival Distributions: Fractional Ages	121
8.1 Uniform distribution of deaths	121
8.2 Constant force of mortality	126
Exercises	127
Solutions	133
9 Survival Distributions: Select Mortality	143
Exercises	147
Solutions	155
10 Survival Distributions: Models for Mortality Improvement	167
10.1 Deterministic mortality improvement models	167
10.2 Stochastic mortality improvement models	170
10.2.1 The Lee Carter model	172
10.2.2 The Cairns-Blake-Dowd (CBD) models	175
Exercises	180
Solutions	184
11 Supplementary Questions: Survival Distributions	191
Solutions	193
II Insurances	199
12 Insurance: Annual and 1/<i>m</i>thly—Moments	201
12.1 Review of Financial Mathematics	201
12.2 Moments of annual insurances	201
12.3 Standard insurances and notation	203
12.4 Standard Ultimate Life Table	205
12.5 Constant force and uniform mortality	206
12.6 Normal approximation	208
12.7 1/ <i>m</i> thly insurance	209
Exercises	210
Solutions	224
13 Insurance: Continuous—Moments—Part 1	237
13.1 Definitions and general formulas	237
13.2 Constant force of mortality	238
Exercises	245
Solutions	254
14 Insurance: Continuous—Moments—Part 2	263
14.1 Uniform survival function	263
14.2 Other mortality functions	265
14.2.1 Integrating $at^n e^{-ct}$ (Gamma Integrands)	265
14.3 Variance of endowment insurance	266

14.4 Normal approximation	268
Exercises	269
Solutions	276
15 Insurance: Probabilities and Percentiles	285
15.1 Introduction	285
15.2 Probabilities for continuous insurance variables	286
15.3 Distribution functions of insurance present values	289
15.4 Probabilities for discrete variables	291
15.5 Percentiles	292
Exercises	295
Solutions	299
16 Insurance: Recursive Formulas, Varying Insurance	307
16.1 Recursive formulas	307
16.2 Varying insurance	309
Exercises	315
Solutions	324
17 Insurance: Relationships between A_x, $A_x^{(m)}$, and \bar{A}_x	335
17.1 Uniform distribution of deaths	335
17.2 Claims acceleration approach	337
Exercises	338
Solutions	341
18 Supplementary Questions: Insurances	345
Solutions	346
III Annuities	349
19 Annuities: Discrete, Expectation	351
19.1 Annuities-due	351
19.2 Annuities-immediate	356
19.3 $1/m$ thly annuities	358
19.4 Actuarial Accumulated Value	359
Exercises	360
Solutions	371
20 Annuities: Continuous, Expectation	381
20.1 Whole life annuity	381
20.2 Temporary and deferred life annuities	384
20.3 n -year certain-and-life annuity	386
Exercises	387
Solutions	394
21 Annuities: Variance	401
21.1 Whole Life and Temporary Life Annuities	401
21.2 Other Annuities	403
21.3 Typical Exam Questions	404
21.4 Combinations of Annuities and Insurances with No Variance	406
Exercises	407
Solutions	416

22 Annuities: Probabilities and Percentiles	431
22.1 Probabilities for continuous annuities	431
22.2 Distribution functions of annuity present values	433
22.3 Probabilities for discrete annuities	434
22.4 Percentiles	436
Exercises	438
Solutions	443
23 Annuities: Varying Annuities, Recursive Formulas	451
23.1 Increasing and Decreasing Annuities	451
23.1.1 Geometrically increasing annuities	451
23.1.2 Arithmetically increasing annuities	451
23.2 Recursive formulas	453
Exercises	455
Solutions	459
24 Annuities: $1/m$-thly Payments	467
24.1 Uniform distribution of deaths assumption	467
24.2 Woolhouse's formula	468
Exercises	472
Solutions	476
25 Supplementary Questions: Annuities	483
Solutions	486
IV Premiums	491
26 Premiums: Net Premiums for Discrete Insurances—Part 1	493
26.1 Future loss	493
26.2 Net premium	494
Exercises	496
Solutions	504
27 Premiums: Net Premiums for Discrete Insurances—Part 2	513
27.1 Premium formulas	513
27.2 Expected value of future loss	514
27.3 International Actuarial Premium Notation	516
Exercises	517
Solutions	524
28 Premiums: Net Premiums Paid on a $1/m$thly Basis	535
Exercises	536
Solutions	539
29 Premiums: Net Premiums for Fully Continuous Insurances	545
Exercises	548
Solutions	554
30 Premiums: Gross Premiums	561
30.1 Gross future loss	561
30.2 Gross premium	562
Exercises	564
Solutions	571

31 Premiums: Variance of Future Loss, Discrete	577
31.1 Variance of net future loss	577
31.1.1 Variance of net future loss by formula	577
31.1.2 Variance of net future loss from first principles	579
31.2 Variance of gross future loss	580
Exercises	582
Solutions	588
32 Premiums: Variance of Future Loss, Continuous	595
32.1 Variance of net future loss	595
32.2 Variance of gross future loss	596
Exercises	597
Solutions	603
33 Premiums: Probabilities and Percentiles of Future Loss	611
33.1 Probabilities	611
33.1.1 Fully continuous insurances	611
33.1.2 Discrete insurances	614
33.1.3 Annuities	615
33.1.4 Gross future loss	617
33.2 Percentiles	618
Exercises	619
Solutions	623
34 Premiums: Special Topics	631
34.1 The portfolio percentile premium principle	631
34.2 Extra risks	633
Exercises	633
Solutions	635
35 Supplementary Questions: Premiums	639
Solutions	642
V Reserves	649
36 Reserves: Net Premium Reserve	651
Exercises	656
Solutions	662
37 Reserves: Gross Premium Reserve and Expense Reserve	671
37.1 Gross premium reserve	671
37.2 Expense reserve	673
Exercises	675
Solutions	678
38 Reserves: Special Formulas for Whole Life and Endowment Insurance	683
38.1 Annuity-ratio formula	683
38.2 Insurance-ratio formula	684
38.3 Premium-ratio formula	685
Exercises	686
Solutions	695
39 Reserves: Variance of Loss	705

Exercises	707
Solutions	713
40 Reserves: Recursive Formulas	721
40.1 Net premium reserve	721
40.2 Insurances and annuities with payment of reserve upon death	724
40.3 Gross premium reserve	727
Exercises	730
Solutions	748
41 Reserves: Modified Reserves	767
Exercises	768
Solutions	772
42 Reserves: Other Topics	779
42.1 Reserves on semicontinuous insurance	779
42.2 Reserves between premium dates	780
42.3 Thiele's differential equation	782
Exercises	785
Solutions	792
43 Supplementary Questions: Reserves	801
Solutions	803
VI Markov Chains	809
44 Markov Chains: Discrete—Probabilities	811
44.1 Introduction	811
44.2 Definition of Markov chains	813
44.3 Discrete Markov chains	815
Exercises	817
Solutions	820
45 Markov Chains: Continuous—Probabilities	825
45.1 Probabilities—direct calculation	826
45.2 Kolmogorov's forward equations	829
Exercises	831
Solutions	839
46 Markov Chains: Premiums and Reserves	845
46.1 Premiums	845
46.2 Reserves	849
46.3 Relationships among insurances and annuities	851
46.3.1 Sum of annuities over all states	851
46.3.2 Temporary annuities and insurances	852
46.3.3 Stepwise calculations	852
46.3.4 Insurance to annuity relationship	853
46.4 Reserve recursions	854
Exercises	856
Solutions	867
47 Applications of Markov Chains	877
47.1 Disability income insurance	877

47.1.1	Features of disability income insurance	877
47.1.2	Premiums and reserves for disability income insurance	878
47.2	Hospital indemnity insurance	880
47.3	Long Term Care Insurance (LTC)	880
47.3.1	Features of LTC	880
47.3.2	Models for LTC	882
47.4	Critical Illness and Chronic Illness Insurance	885
47.5	Continuing care retirement communities (CCRCs)	887
47.5.1	Features of CCRCs	887
47.5.2	Models for CCRCs	887
47.6	Structured settlements	889
Exercises	891
Solutions	904

VII Multiple Decrements 913

48 Multiple Decrement Models: Probabilities	915
48.1 Probabilities	915
48.2 Life tables	917
48.3 Examples of Multiple Decrement Probabilities	918
48.4 Discrete Insurances	919
Exercises	920
Solutions	932

49 Multiple Decrement Models: Forces of Decrement	941
49.1 $\mu_x^{(j)}$	941
49.2 Probability framework for multiple decrement models	943
49.3 Fractional ages	945
Exercises	945
Solutions	954

50 Multiple Decrement Models: Associated Single Decrement Tables	965
Exercises	970
Solutions	974

51 Multiple Decrement Models: Relations Between Rates	983
51.1 Constant force of decrement	983
51.2 Uniform in the multiple-decrement tables	983
51.3 Uniform in the associated single-decrement tables	986
Exercises	990
Solutions	993

52 Multiple Decrement Models: Discrete Decrements	1001
Exercises	1004
Solutions	1009

53 Multiple Decrement Models: Continuous Insurances	1013
Exercises	1016
Solutions	1026

54 Supplementary Questions: Multiple Decrements	1039
Solutions	1040

VIII Multiple Lives	1043
55 Multiple Lives: Joint Life Probabilities	1045
55.1 Markov chain model	1045
55.2 Independent lives	1046
55.3 Joint distribution function model	1048
Exercises	1050
Solutions	1056
56 Multiple Lives: Last Survivor Probabilities	1061
Exercises	1066
Solutions	1071
57 Multiple Lives: Moments	1079
Exercises	1084
Solutions	1088
58 Multiple Lives: Contingent Probabilities	1095
Exercises	1101
Solutions	1107
59 Multiple Lives: Common Shock	1117
Exercises	1119
Solutions	1121
60 Multiple Lives: Insurances	1123
60.1 Joint and last survivor insurances	1123
60.2 Contingent insurances	1128
60.3 Common shock insurances	1129
Exercises	1131
Solutions	1146
61 Multiple Lives: Annuities	1161
61.1 Introduction	1161
61.2 Three techniques for handling annuities	1162
61.3 Variance	1165
Exercises	1166
Solutions	1175
62 Supplementary Questions: Multiple Lives	1185
Solutions	1187
IX Estimating Mortality Rates	1193
63 Review of Mathematical Statistics	1195
63.1 Estimator quality	1195
63.1.1 Bias	1196
63.1.2 Consistency	1197
63.1.3 Variance and mean square error	1198
63.2 Confidence intervals	1200
Exercises	1200
Solutions	1205

64 The Empirical Distribution for Complete Data	1211
64.1 Individual data	1211
64.2 Grouped data	1212
Exercises	1214
Solutions	1217
65 Maximum Likelihood Estimators	1221
65.1 Defining the likelihood	1222
65.1.1 Individual data	1222
65.1.2 Grouped data	1223
65.1.3 Censoring	1224
65.1.4 Truncation	1225
65.1.5 Combination of censoring and truncation	1226
Exercises	1226
Solutions	1234
66 Kaplan-Meier and Nelson-Åalen Estimators	1239
66.1 Kaplan-Meier Estimator	1240
66.2 Nelson-Åalen Estimator	1244
Exercises	1246
Solutions	1255
67 Variance Of Maximum Likelihood Estimators	1265
67.1 Information matrix	1265
67.1.1 Calculating variance using the information matrix	1265
67.1.2 True information and observed information	1268
67.2 The delta method	1270
Exercises	1271
Solutions	1274
68 Variance of Kaplan-Meier and Nelson-Åalen Estimators	1277
68.1 Variance formulas	1277
68.2 Confidence intervals	1279
Exercises	1280
Solutions	1289
69 Mortality Table Construction	1299
69.1 Individual data based methods	1299
69.2 Interval-based methods	1304
69.3 Multiple decrement	1307
69.4 Estimating transition intensities	1309
Exercises	1309
Solutions	1318
 X Pensions and Profit Measures	 1325
70 Pension Mathematics—Basics	1327
70.1 Replacement ratio and salary scale	1327
70.2 Service table	1331
70.3 Actuarial present value of benefits	1332
Exercises	1335
Solutions	1343

71 Pension Mathematics—Valuation	1349
71.1 Actuarial liability for pension benefits	1349
71.2 Funding the benefits	1351
Exercises	1357
Solutions	1362
72 Retiree Health Benefits	1367
72.1 Expected present value	1367
72.2 APBO and Normal Cost	1369
Exercises	1371
Solutions	1375
73 Profit Tests	1379
73.1 Introduction	1379
73.2 Profits by policy year	1380
73.3 Profit measures	1383
73.4 Determining the reserve using a profit test	1386
73.5 Handling multiple-state models	1386
Exercises	1389
Solutions	1395
74 Profit Tests: Gain by Source	1403
Exercises	1407
Solutions	1411
75 Supplementary Questions: Entire Course	1415
Solutions	1431
XI Practice Exams	1451
1 Practice Exam 1	1453
2 Practice Exam 2	1461
3 Practice Exam 3	1473
4 Practice Exam 4	1483
5 Practice Exam 5	1493
6 Practice Exam 6	1503
7 Practice Exam 7	1511
8 Practice Exam 8	1521
9 Practice Exam 9	1531
10 Practice Exam 10	1541
11 Practice Exam 11	1549
12 Practice Exam 12	1559
13 Practice Exam 13	1571

Appendices	1581
A Solutions to the Practice Exams	1583
Solutions for Practice Exam 1	1583
Solutions for Practice Exam 2	1594
Solutions for Practice Exam 3	1604
Solutions for Practice Exam 4	1616
Solutions for Practice Exam 5	1628
Solutions for Practice Exam 6	1639
Solutions for Practice Exam 7	1650
Solutions for Practice Exam 8	1663
Solutions for Practice Exam 9	1674
Solutions for Practice Exam 10	1685
Solutions for Practice Exam 11	1696
Solutions for Practice Exam 12	1707
Solutions for Practice Exam 13	1718
B Solutions to Old Exams	1729
B.1 Solutions to SOA Exam MLC, Spring 2012	1729
B.2 Solutions to SOA Exam MLC, Fall 2012	1736
B.3 Solutions to SOA Exam MLC, Spring 2013	1741
B.4 Solutions to SOA Exam MLC, Fall 2013	1748
B.5 Solutions to SOA Exam MLC, Spring 2014	1755
B.5.1 Multiple choice section	1755
B.5.2 Written answer section	1758
B.6 Solutions to SOA Exam MLC, Fall 2014	1763
B.6.1 Multiple choice section	1763
B.6.2 Written answer section	1767
B.7 Solutions to SOA Exam MLC, Spring 2015	1772
B.7.1 Multiple choice section	1772
B.7.2 Written answer section	1776
B.8 Solutions to SOA Exam MLC, Fall 2015	1781
B.8.1 Multiple choice section	1781
B.8.2 Written answer section	1785
B.9 Solutions to SOA Exam MLC, Spring 2016	1789
B.9.1 Multiple choice section	1789
B.9.2 Written answer section	1794
B.10 Solutions to SOA Exam MLC, Fall 2016	1800
B.10.1 Multiple choice section	1800
B.10.2 Written answer section	1803
B.11 Solutions to SOA Exam MLC, Spring 2017	1808
B.11.1 Multiple choice section	1808
B.11.2 Written answer section	1812
B.12 Solutions to SOA Exam MLC, Fall 2017	1819
B.12.1 Multiple choice section	1819
B.12.2 Written answer section	1822
B.13 Solutions to SOA Exam MLC, Spring 2018	1829
B.13.1 Multiple choice section	1829
B.13.2 Written answer section	1833
B.14 Solutions to SOA Exam LTAM, Fall 2018	1839
B.14.1 Multiple choice section	1839
B.14.2 Written answer section	1842

C Exam Question Index

1849

Lesson 8

Survival Distributions: Fractional Ages

Reading: *Actuarial Mathematics for Life Contingent Risks* 2nd edition 3.2

Life tables list mortality rates (q_x) or lives (l_x) for integral ages only. Often, it is necessary to determine lives at fractional ages (like $l_{x+0.5}$ for x an integer) or mortality rates for fractions of a year. We need some way to interpolate between ages.

8.1 Uniform distribution of deaths

The easiest interpolation method is linear interpolation, or uniform distribution of deaths between integral ages (UDD). This means that the number of lives at age $x + s$, $0 \leq s \leq 1$, is a weighted average of the number of lives at age x and the number of lives at age $x + 1$:

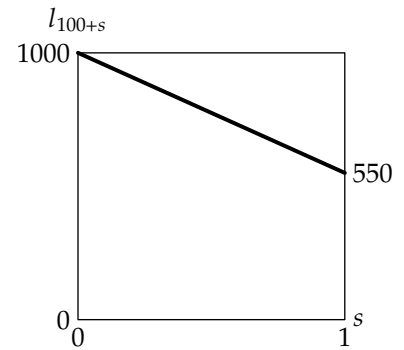
$$l_{x+s} = (1-s)l_x + sl_{x+1} = l_x - sd_x \tag{8.1}$$

The graph of l_{x+s} is a straight line between $s = 0$ and $s = 1$ with slope $-d_x$. The graph at the right portrays this for a mortality rate $q_{100} = 0.45$ and $l_{100} = 1000$.

Contrast UDD with an assumption of a uniform survival function. If age at death is uniformly distributed, then l_x as a function of x is a straight line. If UDD is assumed, l_x is a straight line between integral ages, but the slope may vary for different ages. Thus if age at death is uniformly distributed, UDD holds at all ages, but not conversely.

Using l_{x+s} , we can compute ${}_s q_x$:

$$\begin{aligned} {}_s q_x &= 1 - {}_s p_x \\ &= 1 - \frac{l_{x+s}}{l_x} = 1 - (1 - sq_x) = sq_x \end{aligned} \tag{8.2}$$



That is one of the most important formulas, so let's state it again:

$$\boxed{{}_s q_x = sq_x} \tag{8.2}$$

More generally, for $0 \leq s + t \leq 1$,

$$\begin{aligned} {}_s q_{x+t} &= 1 - {}_s p_{x+t} = 1 - \frac{l_{x+s+t}}{l_{x+t}} \\ &= 1 - \frac{l_x - (s+t)d_x}{l_x - td_x} = \frac{sd_x}{l_x - td_x} = \frac{sq_x}{1 - tq_x} \end{aligned} \tag{8.3}$$

where the last equation was obtained by dividing numerator and denominator by l_x . The important point to pick up is that while ${}_s q_x$ is the proportion of the year s times q_x , the corresponding concept at age $x + t$, ${}_s q_{x+t}$, is *not* sq_x , but is in fact higher than sq_x . The *number* of lives dying in any amount of time is constant, and since there are fewer and fewer lives as the year progresses, the *rate* of death is in fact increasing over the year. The numerator of ${}_s q_{x+t}$ is the proportion of the year being measured s times the death rate, but then this must be divided by 1 minus the proportion of the year that elapsed before the start of measurement.

For most problems involving death probabilities, it will suffice if you remember that l_{x+s} is linearly interpolated. It often helps to create a life table with an arbitrary radix. Try working out the following example before looking at the answer.

EXAMPLE 8A You are given:

- (i) $q_x = 0.1$
- (ii) Uniform distribution of deaths between integral ages is assumed.

Calculate ${}_{1/2}q_{x+1/4}$.

SOLUTION: Let $l_x = 1$. Then $l_{x+1} = l_x(1 - q_x) = 0.9$ and $d_x = 0.1$. Linearly interpolating,

$$\begin{aligned} l_{x+1/4} &= l_x - \frac{1}{4}d_x = 1 - \frac{1}{4}(0.1) = 0.975 \\ l_{x+3/4} &= l_x - \frac{3}{4}d_x = 1 - \frac{3}{4}(0.1) = 0.925 \\ {}_{1/2}q_{x+1/4} &= \frac{l_{x+1/4} - l_{x+3/4}}{l_{x+1/4}} = \frac{0.975 - 0.925}{0.975} = \boxed{0.051282} \end{aligned}$$

You could also use equation (8.3) to work this example. □

EXAMPLE 8B For two lives age (x) with independent future lifetimes, ${}_k|q_x = 0.1(k + 1)$ for $k = 0, 1, 2$. Deaths are uniformly distributed between integral ages.

Calculate the probability that both lives will survive 2.25 years.

SOLUTION: Since the two lives are independent, the probability of both surviving 2.25 years is the square of ${}_{2.25}p_x$, the probability of one surviving 2.25 years. If we let $l_x = 1$ and use $d_{x+k} = l_x {}_k|q_x$, we get

$$\begin{array}{ll} q_x = 0.1(1) = 0.1 & l_{x+1} = 1 - d_x = 1 - 0.1 = 0.9 \\ {}_1|q_x = 0.1(2) = 0.2 & l_{x+2} = 0.9 - d_{x+1} = 0.9 - 0.2 = 0.7 \\ {}_2|q_x = 0.1(3) = 0.3 & l_{x+3} = 0.7 - d_{x+2} = 0.7 - 0.3 = 0.4 \end{array}$$

Then linearly interpolating between l_{x+2} and l_{x+3} , we get

$$\begin{aligned} l_{x+2.25} &= 0.7 - 0.25(0.3) = 0.625 \\ {}_{2.25}p_x &= \frac{l_{x+2.25}}{l_x} = 0.625 \end{aligned}$$

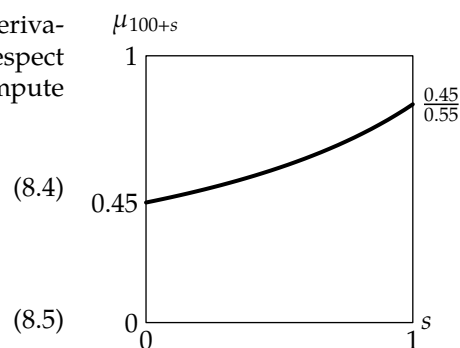
Squaring, the answer is $0.625^2 = \boxed{0.390625}$. □

The probability density function of T_x , ${}_s p_x \mu_{x+s}$, is the constant q_x , the derivative of the conditional cumulative distribution function ${}_s q_x = s q_x$ with respect to s . That is another important formula, since the density is needed to compute expected values, so let's repeat it:

$$\boxed{{}_s p_x \mu_{x+s} = q_x} \tag{8.4}$$

It follows that the force of mortality is q_x divided by $1 - s q_x$:

$$\mu_{x+s} = \frac{q_x}{1 - s q_x} \tag{8.5}$$



The force of mortality increases over the year, as illustrated in the graph for $q_{100} = 0.45$ to the right.



Quiz 8-1 You are given:

- (i) $\mu_{50.4} = 0.01$
- (ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.6}q_{50.4}$.

Complete Expectation of Life Under UDD

Under uniform distribution of deaths between integral ages, if the complete future lifetime random variable T_x is written as $T_x = K_x + R_x$, where K_x is the curtate future lifetime and R_x is the fraction of the last year lived, then K_x and R_x are independent, and R_x is uniform on $[0, 1)$. If uniform distribution of deaths is not assumed, K_x and R_x are usually not independent. Since R_x is uniform on $[0, 1)$, $E[R_x] = \frac{1}{2}$ and $\text{Var}(R_x) = \frac{1}{12}$. It follows from $E[R_x] = \frac{1}{2}$ that

$$\overset{\circ}{e}_x = e_x + \frac{1}{2} \quad (8.6)$$

Let's discuss temporary complete life expectancy. You can always evaluate the temporary complete expectancy, whether or not UDD is assumed, by integrating ${}_t p_x$, as indicated by formula (6.6) on page 78. For UDD, ${}_t p_x$ is linear between integral ages. Therefore, a rule we learned in Lesson 6 applies for all integral x :

$$\overset{\circ}{e}_{x:\overline{t}|} = p_x + 0.5q_x \quad (6.13)$$

This equation will be useful. In addition, the method for generating this equation can be used to work out questions involving temporary complete life expectancies for short periods. The following example illustrates this. This example will be reminiscent of calculating temporary complete life expectancy for uniform mortality.

EXAMPLE 8C You are given

- (i) $q_x = 0.1$.
- (ii) Deaths are uniformly distributed between integral ages.

Calculate $\overset{\circ}{e}_{x:\overline{0.4}|}$.

SOLUTION: We will discuss two ways to solve this: an algebraic method and a geometric method.

The algebraic method is based on the double expectation theorem, equation (1.14). It uses the fact that *for a uniform distribution, the mean is the midpoint*. If deaths occur uniformly between integral ages, then those who die within a period contained within a year survive half the period on the average.

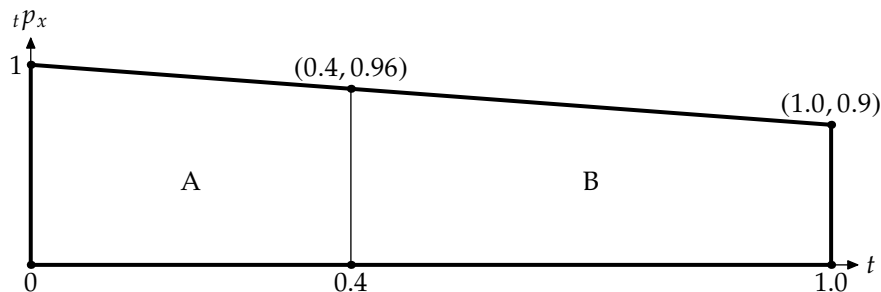
In this example, those who die within 0.4 survive an average of 0.2. Those who survive 0.4 survive an average of 0.4 of course. The temporary life expectancy is the weighted average of these two groups, or $0.4q_x(0.2) + 0.4p_x(0.4)$. This is:

$$0.4q_x = (0.4)(0.1) = 0.04$$

$$0.4p_x = 1 - 0.04 = 0.96$$

$$\overset{\circ}{e}_{x:\overline{0.4}|} = 0.04(0.2) + 0.96(0.4) = \boxed{0.392}$$

An equivalent geometric method, the trapezoidal rule, is to draw the ${}_t p_x$ function from 0 to 0.4. The integral of ${}_t p_x$ is the area under the line, which is the area of a trapezoid: the average of the heights times the width. The following is the graph (not drawn to scale):



Trapezoid A is the area we are interested in. Its area is $\frac{1}{2}(1 + 0.96)(0.4) = \boxed{0.392}$. □



Quiz 8-2 As in Example 8C, you are given

- (i) $q_x = 0.1$.
- (ii) Deaths are uniformly distributed between integral ages.

Calculate $\dot{e}_{x+0.4:\overline{0.6}|}$.

Let's now work out an example in which the duration crosses an integral boundary.

EXAMPLE 8D You are given:

- (i) $q_x = 0.1$
- (ii) $q_{x+1} = 0.2$
- (iii) Deaths are uniformly distributed between integral ages.

Calculate $\dot{e}_{x+0.5:\overline{1}|}$.

SOLUTION: Let's start with the algebraic method. Since the mortality rate changes at $x + 1$, we must split the group into those who die before $x + 1$, those who die afterwards, and those who survive. Those who die before $x + 1$ live 0.25 on the average since the period to $x + 1$ is length 0.5. Those who die after $x + 1$ live between 0.5 and 1 years; the midpoint of 0.5 and 1 is 0.75, so they live 0.75 years on the average. Those who survive live 1 year.

Now let's calculate the probabilities.

$$\begin{aligned} {}_{0.5}q_{x+0.5} &= \frac{0.5(0.1)}{1 - 0.5(0.1)} = \frac{5}{95} \\ {}_{0.5}p_{x+0.5} &= 1 - \frac{5}{95} = \frac{90}{95} \\ {}_{0.5|0.5}q_{x+0.5} &= \left(\frac{90}{95}\right)(0.5(0.2)) = \frac{9}{95} \\ {}_1p_{x+0.5} &= 1 - \frac{5}{95} - \frac{9}{95} = \frac{81}{95} \end{aligned}$$

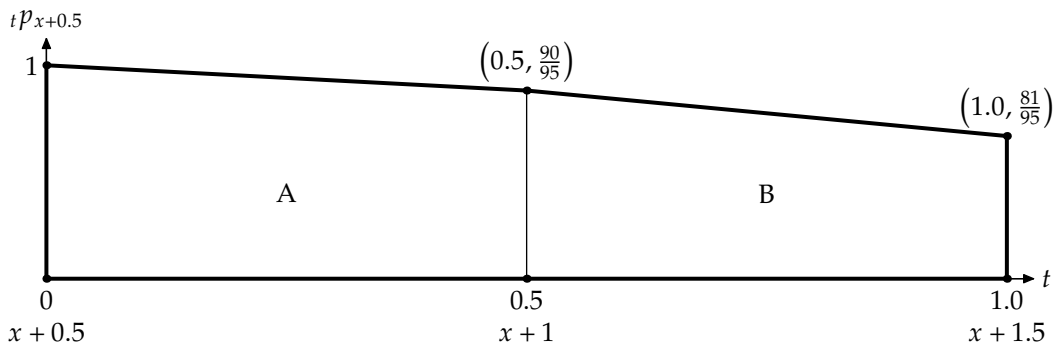
These probabilities could also be calculated by setting up an l_x table with radix 100 at age x and interpolating within it to get $l_{x+0.5}$ and $l_{x+1.5}$. Then

$$\begin{aligned} l_{x+1} &= 0.9l_x = 90 \\ l_{x+2} &= 0.8l_{x+1} = 72 \\ l_{x+0.5} &= 0.5(90 + 100) = 95 \\ l_{x+1.5} &= 0.5(72 + 90) = 81 \\ {}_{0.5}q_{x+0.5} &= 1 - \frac{90}{95} = \frac{5}{95} \\ {}_{0.5|0.5}q_{x+0.5} &= \frac{90 - 81}{95} = \frac{9}{95} \\ {}_1p_{x+0.5} &= \frac{l_{x+1.5}}{l_{x+0.5}} = \frac{81}{95} \end{aligned}$$

Either way, we're now ready to calculate $\dot{e}_{x+0.5:\overline{1}|}$.

$$\dot{e}_{x+0.5:\overline{1}|} = \frac{5(0.25) + 9(0.75) + 81(1)}{95} = \boxed{\frac{89}{95}}$$

For the geometric method we draw the following graph:



The heights at $x+1$ and $x+1.5$ are as we computed above. Then we compute each area separately. The area of A is $\frac{1}{2} \left(1 + \frac{90}{95}\right) (0.5) = \frac{185}{95(4)}$. The area of B is $\frac{1}{2} \left(\frac{90}{95} + \frac{81}{95}\right) (0.5) = \frac{171}{95(4)}$. Adding them up, we get $\frac{185+171}{95(4)} = \frac{89}{95}$. \square



Quiz 8-3 The probability that a battery fails by the end of the k^{th} month is given in the following table:

k	Probability of battery failure by the end of month k
1	0.05
2	0.20
3	0.60

Between integral months, time of failure for the battery is uniformly distributed. Calculate the expected amount of time the battery survives within 2.25 months.

To calculate $\dot{e}_{x:\overline{n}|}$ in terms of $e_{x:\overline{n}|}$ when x and n are both integers, note that those who survive n years contribute the same to both. Those who die contribute an average of $\frac{1}{2}$ more to $\dot{e}_{x:\overline{n}|}$ since they die on the average in the middle of the year. Thus the difference is $\frac{1}{2}nq_x$:

$$\dot{e}_{x:\overline{n}|} = e_{x:\overline{n}|} + 0.5nq_x \quad (8.7)$$

EXAMPLE 8E You are given:

- (i) $q_x = 0.01$ for $x = 50, 51, \dots, 59$.
- (ii) Deaths are uniformly distributed between integral ages.

Calculate $\dot{e}_{50:\overline{10}|}$.

SOLUTION: As we just said, $\dot{e}_{50:\overline{10}|} = e_{50:\overline{10}|} + 0.5_{10}q_{50}$. The first summand, $e_{50:\overline{10}|}$, is the sum of ${}_k p_{50} = 0.99^k$ for $k = 1, \dots, 10$. This sum is a geometric series:

$$e_{50:\overline{10}|} = \sum_{k=1}^{10} 0.99^k = \frac{0.99 - 0.99^{11}}{1 - 0.99} = 9.46617$$

The second summand, the probability of dying within 10 years is ${}_{10}q_{50} = 1 - 0.99^{10} = 0.095618$. Therefore

$$\dot{e}_{50:\overline{10}|} = 9.46617 + 0.5(0.095618) = \boxed{9.51398} \quad \square$$

8.2 Constant force of mortality

The constant force of mortality interpolation method sets μ_{x+s} equal to a constant for x an integral age and $0 < s \leq 1$. Since $p_x = \exp\left(-\int_0^1 \mu_{x+s} ds\right)$ and $\mu_{x+s} = \mu$ is constant,

$$p_x = e^{-\mu} \quad (8.8)$$

$$\mu = -\ln p_x \quad (8.9)$$

Therefore

$${}_s p_x = e^{-\mu s} = (p_x)^s \quad (8.10)$$

In fact, ${}_s p_{x+t}$ is independent of t for $0 \leq t \leq 1 - s$.

$${}_s p_{x+t} = (p_x)^s \quad (8.11)$$

for any $0 \leq t \leq 1 - s$. Figure 8.1 shows l_{100+s} and μ_{100+s} for $l_{100} = 1000$ and $q_{100} = 0.45$ if constant force of mortality is assumed.

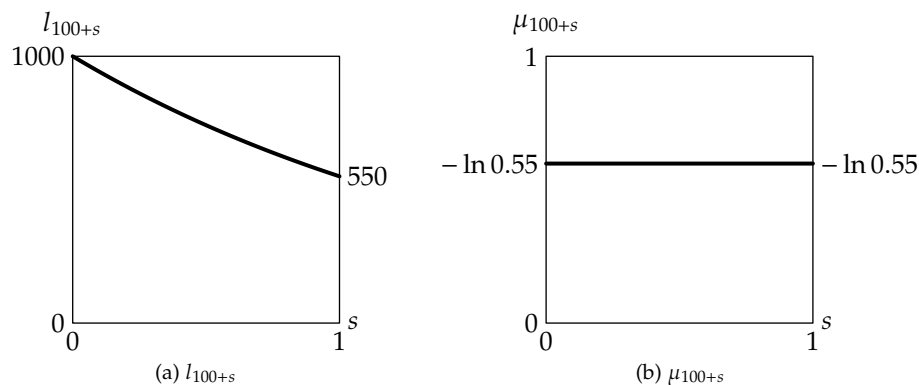


Figure 8.1: Example of constant force of mortality

Contrast constant force of mortality between integral ages to global constant force of mortality, which was introduced in Subsection 5.2.1. The method discussed here allows μ_x to vary for different integers x .

We will now repeat some of the earlier examples but using constant force of mortality.

EXAMPLE 8F You are given:

- (i) $q_x = 0.1$
- (ii) The force of mortality is constant between integral ages.

Calculate ${}_{1/2}q_{x+1/4}$.

SOLUTION:

$${}_{1/2}q_{x+1/4} = 1 - {}_{1/2}p_{x+1/4} = 1 - p_x^{1/2} = 1 - 0.9^{1/2} = 1 - 0.948683 = \boxed{0.051317} \quad \square$$

EXAMPLE 8G You are given:

- (i) $q_x = 0.1$
- (ii) $q_{x+1} = 0.2$
- (iii) The force of mortality is constant between integral ages.

Calculate $\dot{e}_{x+0.5:\overline{1}|}$.

Table 8.1: Summary of formulas for fractional ages

Function	Uniform distribution of deaths	Constant force of mortality
l_{x+s}	$l_x - s d_x$	$l_x p_x^s$
${}_s q_x$	$s q_x$	$1 - p_x^s$
${}_s p_x$	$1 - s q_x$	p_x^s
${}_s q_{x+t}$	$s q_x / (1 - t q_x)$	$1 - p_x^s$
μ_{x+s}	$q_x / (1 - s q_x)$	$-\ln p_x$
${}_s p_x \mu_{x+s}$	q_x	$-p_x^s \ln p_x$
\dot{e}_x	$e_x + 0.5$	
$\dot{e}_{x:\overline{n} }$	$e_{x:\overline{n} } + 0.5 {}_n q_x$	
$\dot{e}_{x:\overline{1} }$	$p_x + 0.5 q_x$	

SOLUTION: We calculate $\int_0^1 {}_t p_{x+0.5} dt$. We split this up into two integrals, one from 0 to 0.5 for age x and one from 0.5 to 1 for age $x + 1$. The first integral is

$$\int_0^{0.5} {}_t p_{x+0.5} dt = \int_0^{0.5} p_x^t dt = \int_0^{0.5} 0.9^t dt = -\frac{1 - 0.9^{0.5}}{\ln 0.9} = 0.487058$$

For $t > 0.5$,

$${}_t p_{x+0.5} = 0.5 p_{x+0.5} {}_{t-0.5} p_{x+1} = 0.9^{0.5} {}_{t-0.5} p_{x+1}$$

so the second integral is

$$0.9^{0.5} \int_{0.5}^1 {}_{t-0.5} p_{x+1} dt = 0.9^{0.5} \int_0^{0.5} 0.8^t dt = -(0.9^{0.5}) \left(\frac{1 - 0.8^{0.5}}{\ln 0.8} \right) = (0.948683)(0.473116) = 0.448837$$

The answer is $\dot{e}_{x+0.5:\overline{1}|} = 0.487058 + 0.448837 = \boxed{0.935895}$. □

Although constant force of mortality is not used as often as UDD, it can be useful for simplifying formulas under certain circumstances. Calculating the expected present value of an insurance where the death benefit within a year follows an exponential pattern (this can happen when the death benefit is the discounted present value of something) may be easier with constant force of mortality than with UDD. The formulas for this lesson are summarized in Table 8.1.

Exercises

Uniform distribution of death

8.1. [CAS4-S85:16] (1 point) Deaths are uniformly distributed between integral ages.

Which of the following represents ${}_3/4 p_x + \frac{1}{2} {}_1/2 p_x \mu_{x+1/2}$?

- (A) ${}_3/4 p_x$ (B) ${}_3/4 q_x$ (C) ${}_1/2 p_x$ (D) ${}_1/2 q_x$ (E) ${}_1/4 p_x$

8.2. [Based on 150-S88:25] You are given:

- (i) $0.25 q_{x+0.75} = 3/31$.
(ii) Mortality is uniformly distributed within age x .

Calculate q_x .

Use the following information for questions 8.3 and 8.4:

You are given:

- (i) Deaths are uniformly distributed between integral ages.
- (ii) $q_x = 0.10$.
- (iii) $q_{x+1} = 0.15$.

8.3. Calculate ${}_{1/2}q_{x+3/4}$.

8.4. Calculate ${}_{0.3|0.5}q_{x+0.4}$.

8.5. You are given:

- (i) Deaths are uniformly distributed between integral ages.
- (ii) Mortality follows the Standard Ultimate Life Table.

Calculate the median future lifetime for (45.5).

8.6. [160-F90:5] You are given:

- (i) A survival distribution is defined by

$$l_x = 1000 \left(1 - \left(\frac{x}{100} \right)^2 \right), \quad 0 \leq x \leq 100.$$

- (ii) μ_x denotes the actual force of mortality for the survival distribution.
- (iii) μ_x^L denotes the approximation of the force of mortality based on the uniform distribution of deaths assumption for l_x , $50 \leq x < 51$.

Calculate $\mu_{50.25} - \mu_{50.25}^L$.

- (A) -0.00016 (B) -0.00007 (C) 0 (D) 0.00007 (E) 0.00016

8.7. A survival distribution is defined by

- (i) $S_0(k) = 1/(1 + 0.01k)^4$ for k a non-negative integer.
- (ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.4}q_{20.2}$.

8.8. [Based on 150-S89:15] You are given:

(i) Deaths are uniformly distributed over each year of age.

(ii)	x	l_x
	35	100
	36	99
	37	96
	38	92
	39	87

Which of the following are true?

- I. ${}_{1|2}q_{36} = 0.091$
 II. $\mu_{37.5} = 0.043$
 III. ${}_{0.33}q_{38.5} = 0.021$

- (A) I and II only (B) I and III only (C) II and III only (D) I, II and III
 (E) The correct answer is not given by (A), (B), (C), or (D).

8.9. [150-82-94:5] You are given:

(i) Deaths are uniformly distributed over each year of age.

(ii) ${}_{0.75}p_x = 0.25$.

Which of the following are true?

- I. ${}_{0.25}q_{x+0.5} = 0.5$
 II. ${}_{0.5}q_x = 0.5$
 III. $\mu_{x+0.5} = 0.5$

- (A) I and II only (B) I and III only (C) II and III only (D) I, II and III
 (E) The correct answer is not given by (A), (B), (C), or (D).

8.10. [3-S00:12] For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$

(ii) $\mu_{81.5} = 0.0408$

(iii) $\mu_{82.5} = 0.0619$

(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

- (A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

8.11. You are given:

(i) Deaths are uniformly distributed between integral ages.

(ii) $q_x = 0.1$.

(iii) $q_{x+1} = 0.3$.

Calculate $\dot{e}_{x+0.7:\overline{1}|}$.

8.12. You are given:

- (i) Deaths are uniformly distributed between integral ages.
- (ii) $q_{45} = 0.01$.
- (iii) $q_{46} = 0.011$.

Calculate $\text{Var}(\min(T_{45}, 2))$.

8.13. You are given:

- (i) Deaths are uniformly distributed between integral ages.
- (ii) ${}_{10}p_x = 0.2$.

Calculate $\dot{e}_{x:\overline{10}|} - e_{x:\overline{10}|}$.

8.14. [4-F86:21] You are given:

- (i) $q_{60} = 0.020$
- (ii) $q_{61} = 0.022$
- (iii) Deaths are uniformly distributed over each year of age.

Calculate $\dot{e}_{60:\overline{1.5}|}$.

- (A) 1.447 (B) 1.457 (C) 1.467 (D) 1.477 (E) 1.487

8.15. [150-F89:21] You are given:

- (i) $q_{70} = 0.040$
- (ii) $q_{71} = 0.044$
- (iii) Deaths are uniformly distributed over each year of age.

Calculate $\dot{e}_{70:\overline{1.5}|}$.

- (A) 1.435 (B) 1.445 (C) 1.455 (D) 1.465 (E) 1.475

8.16. [3-S01:33] For a 4-year college, you are given the following probabilities for dropout from all causes:

$$\begin{aligned} q_0 &= 0.15 \\ q_1 &= 0.10 \\ q_2 &= 0.05 \\ q_3 &= 0.01 \end{aligned}$$

Dropouts are uniformly distributed over each year.

Compute the temporary 1.5-year complete expected college lifetime of a student entering the second year, $\dot{e}_{1:\overline{1.5}|}$.

- (A) 1.25 (B) 1.30 (C) 1.35 (D) 1.40 (E) 1.45

8.17. You are given:

- (i) Deaths are uniformly distributed between integral ages.
- (ii) $\dot{e}_{x+0.5:\overline{0.5}|} = 5/12$.

Calculate q_x .

8.18. You are given:

- (i) Deaths are uniformly distributed over each year of age.
- (ii) $\dot{e}_{55:\overline{20}|} = 0.396$.

Calculate $\mu_{55.2}$.

8.19. [150-S87:21] You are given:

- (i) $d_x = k$ for $x = 0, 1, 2, \dots, \omega - 1$
- (ii) $\dot{e}_{20:\overline{20}|} = 18$
- (iii) Deaths are uniformly distributed over each year of age.

Calculate ${}_{30|10}q_{30}$.

- (A) 0.111 (B) 0.125 (C) 0.143 (D) 0.167 (E) 0.200

8.20. [150-S89:24] You are given:

- (i) Deaths are uniformly distributed over each year of age.
- (ii) $\mu_{45.5} = 0.5$

Calculate $\dot{e}_{45:\overline{1}|}$.

- (A) 0.4 (B) 0.5 (C) 0.6 (D) 0.7 (E) 0.8

8.21. [CAS3-S04:10] 4,000 people age (30) each pay an amount, P , into a fund. Immediately after the 1,000th death, the fund will be dissolved and each of the survivors will be paid \$50,000.

- Mortality follows the Standard Ultimate Life Table, using linear interpolation at fractional ages.
- $i = 12\%$

Calculate P .

Constant force of mortality

8.22. [160-F87:5] Based on given values of l_x and l_{x+1} , ${}_{1/4}p_{x+1/4} = 49/50$ under the assumption of constant force of mortality.

Calculate ${}_{1/4}p_{x+1/4}$ under the uniform distribution of deaths hypothesis.

- (A) 0.9799 (B) 0.9800 (C) 0.9801 (D) 0.9802 (E) 0.9803

8.23. [160-S89:5] A mortality study is conducted for the age interval $(x, x + 1]$.

If a constant force of mortality applies over the interval, ${}_{0.25}q_{x+0.1} = 0.05$.

Calculate ${}_{0.25}q_{x+0.1}$ assuming a uniform distribution of deaths applies over the interval.

- (A) 0.044 (B) 0.047 (C) 0.050 (D) 0.053 (E) 0.056

8.24. [150-F89:29] You are given that $q_x = 0.25$.

Based on the constant force of mortality assumption, the force of mortality is μ_{x+s}^A , $0 < s < 1$.

Based on the uniform distribution of deaths assumption, the force of mortality is μ_{x+s}^B , $0 < s < 1$.

Calculate the smallest s such that $\mu_{x+s}^B \geq \mu_{x+s}^A$.

- (A) 0.4523 (B) 0.4758 (C) 0.5001 (D) 0.5239 (E) 0.5477

8.25. [160-S91:4] From a population mortality study, you are given:

(i) Within each age interval, $[x + k, x + k + 1)$, the force of mortality, μ_{x+k} , is constant.

(ii) k	$e^{-\mu_{x+k}}$	$\frac{1 - e^{-\mu_{x+k}}}{\mu_{x+k}}$
0	0.98	0.99
1	0.96	0.98

Calculate $\overset{\circ}{e}_{x:\overline{2}|}$, the expected lifetime in years over $(x, x + 2]$.

- (A) 1.92 (B) 1.94 (C) 1.95 (D) 1.96 (E) 1.97

8.26. You are given:

(i) $q_{80} = 0.1$

(ii) $q_{81} = 0.2$

(iii) The force of mortality is constant between integral ages.

Calculate $\overset{\circ}{e}_{80.4:\overline{0.8}|}$.

8.27. [3-S01:27] An actuary is modeling the mortality of a group of 1000 people, each age 95, for the next three years.

The actuary starts by calculating the expected number of survivors at each integral age by

$$l_{95+k} = 1000 {}_k p_{95}, \quad k = 1, 2, 3$$

The actuary subsequently calculates the expected number of survivors at the middle of each year using the assumption that deaths are uniformly distributed over each year of age.

This is the result of the actuary's model:

Age	Survivors
95	1000
95.5	800
96	600
96.5	480
97	—
97.5	288
98	—

The actuary decides to change his assumption for mortality at fractional ages to the constant force assumption. He retains his original assumption for each ${}_k p_{95}$.

Calculate the revised expected number of survivors at age 97.5.

- (A) 270 (B) 273 (C) 276 (D) 279 (E) 282

8.28. [M-F06:16] You are given the following information on participants entering a 2-year program for treatment of a disease:

(i) Only 10% survive to the end of the second year.

(ii) The force of mortality is constant within each year.

(iii) The force of mortality for year 2 is three times the force of mortality for year 1.

Calculate the probability that a participant who survives to the end of month 3 dies by the end of month 21.

- (A) 0.61 (B) 0.66 (C) 0.71 (D) 0.75 (E) 0.82

8.29. You are given:

(i) $\mu_x = \sqrt{\frac{1}{80-x}}, \quad 0 \leq x \leq 80$

(ii) F is the exact value of $S_0(10.5)$.

(iii) G is the value of $S_0(10.5)$ using the constant force assumption for interpolation between ages 10 and 11.

Calculate $F - G$.

- (A) -0.01083 (B) -0.00005 (C) 0 (D) 0.00003 (E) 0.00172

Additional old SOA Exam MLC questions: S12:2, F13:25, F16:1, S18:2

Additional old CAS Exam 3/3L questions: S05:31, F05:13, S06:13, F06:13, S07:24, S08:16, S09:3, F09:3, S10:4, F10:3, S11:3, S12:3, F12:3, S13:3, F13:3

Additional old CAS Exam LC questions: S14:4, F14:4, S15:3, F15:3

Multiple choice sample questions: 3.5, 3.11, 4.7

Solutions

8.1. In the second summand, ${}_{1/2}p_x \mu_{x+1/2}$ is the density function, which is the constant q_x under UDD. The first summand ${}_{3/4}p_x = 1 - \frac{3}{4}q_x$. So the sum is $1 - \frac{1}{4}q_x$, or $\boxed{{}_{1/4}p_x}$. (E)

8.2. Using equation (8.3),

$$\begin{aligned} \frac{3}{31} &= {}_{0.25}q_{x+0.75} = \frac{0.25q_x}{1-0.75q_x} \\ \frac{3}{31} - \frac{2.25}{31}q_x &= 0.25q_x \\ \frac{3}{31} &= \frac{10}{31}q_x \\ q_x &= \boxed{0.3} \end{aligned}$$

8.3. We calculate the probability that $(x + \frac{3}{4})$ survives for half a year. Since the duration crosses an integer boundary, we break the period up into two quarters of a year. The probability of $(x + 3/4)$ surviving for 0.25 years is, by equation (8.3),

$${}_{1/4}p_{x+3/4} = \frac{1-0.10}{1-0.75(0.10)} = \frac{0.9}{0.925}$$

The probability of $(x + 1)$ surviving to $x + 1.25$ is

$${}_{1/4}p_{x+1} = 1 - 0.25(0.15) = 0.9625$$

The answer to the question is then the complement of the product of these two numbers:

$${}_{1/2}q_{x+3/4} = 1 - {}_{1/2}p_{x+3/4} = 1 - {}_{1/4}p_{x+3/4} {}_{1/4}p_{x+1} = 1 - \left(\frac{0.9}{0.925}\right)(0.9625) = \boxed{0.06351}$$

Alternatively, you could build a life table starting at age x , with $l_x = 1$. Then $l_{x+1} = (1 - 0.1) = 0.9$ and $l_{x+2} = 0.9(1 - 0.15) = 0.765$. Under UDD, l_x at fractional ages is obtained by linear interpolation, so

$$\begin{aligned} l_{x+0.75} &= 0.75(0.9) + 0.25(1) = 0.925 \\ l_{x+1.25} &= 0.25(0.765) + 0.75(0.9) = 0.86625 \\ {}_{1/2}p_{3/4} &= \frac{l_{x+1.25}}{l_{x+0.75}} = \frac{0.86625}{0.925} = 0.93649 \\ {}_{1/2}q_{3/4} &= 1 - {}_{1/2}p_{3/4} = 1 - 0.93649 = \boxed{0.06351} \end{aligned}$$

8.4. ${}_{0.3|0.5}q_{x+0.4}$ is ${}_{0.3}p_{x+0.4} - {}_{0.8}p_{x+0.4}$. The first summand is

$${}_{0.3}p_{x+0.4} = \frac{1 - 0.7q_x}{1 - 0.4q_x} = \frac{1 - 0.07}{1 - 0.04} = \frac{93}{96}$$

The probability that $(x + 0.4)$ survives to $x + 1$ is, by equation (8.3),

$${}_{0.6}p_{x+0.4} = \frac{1 - 0.10}{1 - 0.04} = \frac{90}{96}$$

and the probability $(x + 1)$ survives to $x + 1.2$ is

$${}_{0.2}p_{x+1} = 1 - 0.2q_{x+1} = 1 - 0.2(0.15) = 0.97$$

So

$${}_{0.3|0.5}q_{x+0.4} = \frac{93}{96} - \left(\frac{90}{96}\right)(0.97) = \boxed{0.059375}$$

Alternatively, you could use the life table from the solution to the last question, and linearly interpolate:

$$\begin{aligned} l_{x+0.4} &= 0.4(0.9) + 0.6(1) = 0.96 \\ l_{x+0.7} &= 0.7(0.9) + 0.3(1) = 0.93 \\ l_{x+1.2} &= 0.2(0.765) + 0.8(0.9) = 0.873 \\ {}_{0.3|0.5}q_{x+0.4} &= \frac{0.93 - 0.873}{0.96} = \boxed{0.059375} \end{aligned}$$

8.5. Under uniform distribution of deaths between integral ages, $l_{x+0.5} = \frac{1}{2}(l_x + l_{x+1})$, since the survival function is a straight line between two integral ages. Therefore, $l_{45.5} = \frac{1}{2}(99,033.9 + 98,957.6) = 98,995.75$. Median future lifetime occurs when $l_x = \frac{1}{2}(98,995.75) = 49,497.9$. This happens between ages 88 and 89. We interpolate between the ages to get the exact median:

$$\begin{aligned} l_{88} - s(l_{88} - l_{89}) &= 49,497.9 \\ 50,038.6 - s(50,038.6 - 45,995.6) &= 49,497.9 \\ 50,038.6 - 4,043.0s &= 49,497.9 \\ s &= \frac{50,038.6 - 49,497.9}{4,043.0} = 0.13374 \end{aligned}$$

So the median age at death is 88.13374, and median future lifetime is $88.13374 - 45.5 = \boxed{42.63374}$.

8.6. ${}_x p_0 = \frac{l_x}{l_0} = 1 - \left(\frac{x}{100}\right)^2$. The force of mortality is calculated as the negative derivative of $\ln {}_x p_0$:

$$\begin{aligned} \mu_x &= -\frac{d \ln {}_x p_0}{dx} = \frac{2\left(\frac{x}{100}\right)\left(\frac{1}{100}\right)}{1 - \left(\frac{x}{100}\right)^2} = \frac{2x}{100^2 - x^2} \\ \mu_{50.25} &= \frac{100.5}{100^2 - 50.25^2} = 0.0134449 \end{aligned}$$

For UDD, we need to calculate q_{50} .

$$\begin{aligned} p_{50} &= \frac{l_{51}}{l_{50}} = \frac{1 - 0.51^2}{1 - 0.50^2} = 0.986533 \\ q_{50} &= 1 - 0.986533 = 0.013467 \end{aligned}$$

so under UDD,

$$\mu_{50.25}^L = \frac{q_{50}}{1 - 0.25q_{50}} = \frac{0.013467}{1 - 0.25(0.013467)} = 0.013512.$$

The difference between $\mu_{50.25}$ and $\mu_{50.25}^L$ is $0.013445 - 0.013512 = \boxed{-0.000067}$. (B)

8.7. $S_0(20) = 1/1.2^4$ and $S_0(21) = 1/1.21^4$, so $q_{20} = 1 - (1.2/1.21)^4 = 0.03265$. Then

$${}_{0.4}q_{20.2} = \frac{0.4q_{20}}{1 - 0.2q_{20}} = \frac{0.4(0.03265)}{1 - 0.2(0.03265)} = \boxed{0.01315}$$

8.8.

I. Calculate ${}_{1|2}q_{36}$.

$${}_{1|2}q_{36} = \frac{{}_2d_{37}}{l_{36}} = \frac{96 - 87}{99} = 0.09091 \quad \checkmark$$

This statement does not require uniform distribution of deaths.

II. By equation (8.5),

$$\mu_{37.5} = \frac{q_{37}}{1 - 0.5q_{37}} = \frac{4/96}{1 - 2/96} = \frac{4}{94} = 0.042553 \quad \checkmark$$

III. Calculate ${}_{0.33}q_{38.5}$.

$${}_{0.33}q_{38.5} = \frac{0.33d_{38.5}}{l_{38.5}} = \frac{(0.33)(5)}{89.5} = 0.018436 \quad \times$$

I can't figure out what mistake you'd have to make to get 0.021. (A)

8.9. First calculate q_x .

$$\begin{aligned} 1 - 0.75q_x &= 0.25 \\ q_x &= 1 \end{aligned}$$

Then by equation (8.3), ${}_{0.25}q_{x+0.5} = 0.25/(1 - 0.5) = 0.5$, making I true.

By equation (8.2), ${}_{0.5}q_x = 0.5q_x = 0.5$, making II true.

By equation (8.5), $\mu_{x+0.5} = 1/(1 - 0.5) = 2$, making III false. (A)

8.10. We use equation (8.5) to back out q_x for each age.

$$\begin{aligned} \mu_{x+0.5} &= \frac{q_x}{1 - 0.5q_x} \Rightarrow q_x = \frac{\mu_{x+0.5}}{1 + 0.5\mu_{x+0.5}} \\ q_{80} &= \frac{0.0202}{1.0101} = 0.02 \\ q_{81} &= \frac{0.0408}{1.0204} = 0.04 \\ q_{82} &= \frac{0.0619}{1.03095} = 0.06 \end{aligned}$$

Then by equation (8.3), ${}_{0.5}p_{80.5} = 0.98/0.99$, $p_{81} = 0.96$, and ${}_{0.5}p_{82} = 1 - 0.5(0.06) = 0.97$. Therefore

$${}_{2}q_{80.5} = 1 - \left(\frac{0.98}{0.99}\right)(0.96)(0.97) = \boxed{0.0782} \quad (\text{A})$$

8.11. To do this algebraically, we split the group into those who die within 0.3 years, those who die between 0.3 and 1 year, and those who survive one year. Under UDD, those who die will die at the midpoint of the interval (assuming the interval doesn't cross an integral age), so we have

Group	Survival time	Probability of group	Average survival time
I	(0, 0.3]	$1 - {}_{0.3}p_{x+0.7}$	0.15
II	(0.3, 1]	${}_{0.3}p_{x+0.7} - {}_1p_{x+0.7}$	0.65
III	(1, ∞)	${}_1p_{x+0.7}$	1

We calculate the required probabilities.

$$\begin{aligned} {}_{0.3}p_{x+0.7} &= \frac{0.9}{0.93} = 0.967742 \\ {}_1p_{x+0.7} &= \frac{0.9}{0.93}(1 - 0.7(0.3)) = 0.764516 \\ 1 - {}_{0.3}p_{x+0.7} &= 1 - 0.967742 = 0.032258 \\ {}_{0.3}p_{x+0.7} - {}_1p_{x+0.7} &= 0.967742 - 0.764516 = 0.203226 \\ \dot{e}_{x+0.7:\overline{1}} &= 0.032258(0.15) + 0.203226(0.65) + 0.764516(1) = \boxed{0.901452} \end{aligned}$$

Alternatively, we can use trapezoids. We already know from the above solution that the heights of the first trapezoid are 1 and 0.967742, and the heights of the second trapezoid are 0.967742 and 0.764516. So the sum of the area of the two trapezoids is

$$\begin{aligned} \dot{e}_{x+0.7:\overline{1}} &= (0.3)(0.5)(1 + 0.967742) + (0.7)(0.5)(0.967742 + 0.764516) \\ &= 0.295161 + 0.606290 = \boxed{0.901451} \end{aligned}$$

8.12. For the expected value, we'll use the recursive formula. (The trapezoidal rule could also be used.)

$$\begin{aligned} \dot{e}_{45:\overline{2}} &= \dot{e}_{45:\overline{1}} + p_{45} \dot{e}_{46:\overline{1}} \\ &= (1 - 0.005) + 0.99(1 - 0.0055) \\ &= 1.979555 \end{aligned}$$

We'll use equation (6.7) to calculate the second moment.

$$\begin{aligned} E[\min(T_{45}, 2)^2] &= 2 \int_0^2 t {}_t p_x dt \\ &= 2 \left(\int_0^1 t(1 - 0.01t) dt + \int_1^2 t(0.99)(1 - 0.011(t - 1)) dt \right) \\ &= 2 \left(\frac{1}{2} - 0.01 \left(\frac{1}{3} \right) + 0.99 \left(\frac{(1.011)(2^2 - 1^2)}{2} - 0.011 \left(\frac{2^3 - 1^3}{3} \right) \right) \right) \\ &= 2(0.496667 + 1.475925) = 3.94518 \end{aligned}$$

So the variance is $3.94518 - 1.979555^2 = \boxed{0.02654}$.

8.13. As discussed on page 125, by equation (8.7), the difference is

$$\frac{1}{2} {}_{10}q_x = \frac{1}{2}(1 - 0.2) = \boxed{0.4}$$

8.14. Those who die in the first year survive $\frac{1}{2}$ year on the average and those who die in the first half of the second year survive 1.25 years on the average, so we have

$$\begin{aligned} p_{60} &= 0.98 \\ {}_{1.5}p_{60} &= 0.98(1 - 0.5(0.022)) = 0.96922 \\ \dot{e}_{60:\overline{1.5}} &= 0.5(0.02) + 1.25(0.98 - 0.96922) + 1.5(0.96922) = \boxed{1.477305} \quad (\text{D}) \end{aligned}$$

Alternatively, we use the trapezoidal method. The first trapezoid has heights 1 and $p_{60} = 0.98$ and width 1. The second trapezoid has heights $p_{60} = 0.98$ and ${}_{1.5}p_{60} = 0.96922$ and width $1/2$.

$$\begin{aligned} \dot{e}_{60:\overline{1.5}} &= \frac{1}{2}(1 + 0.98) + \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) (0.98 + 0.96922) \\ &= \boxed{1.477305} \quad (\text{D}) \end{aligned}$$

8.15. $p_{70} = 1 - 0.040 = 0.96$, ${}_2p_{70} = (0.96)(0.956) = 0.91776$, and by linear interpolation, ${}_{1.5}p_{70} = 0.5(0.96 + 0.91776) = 0.93888$. Those who die in the first year survive 0.5 years on the average and those who die in the first half of the second year survive 1.25 years on the average. So

$$\dot{e}_{70:\overline{1.5}|} = 0.5(0.04) + 1.25(0.96 - 0.93888) + 1.5(0.93888) = \boxed{1.45472} \quad (\text{C})$$

Alternatively, we can use the trapezoidal method. The first year's trapezoid has heights 1 and 0.96 and width 1 and the second year's trapezoid has heights 0.96 and 0.93888 and width $1/2$, so

$$\dot{e}_{70:\overline{1.5}|} = 0.5(1 + 0.96) + 0.5(0.5)(0.96 + 0.93888) = \boxed{1.45472} \quad (\text{C})$$

8.16. First we calculate ${}_t p_1$ for $t = 1, 2$.

$$p_1 = 1 - q_1 = 0.90$$

$${}_2p_1 = (1 - q_1)(1 - q_2) = (0.90)(0.95) = 0.855$$

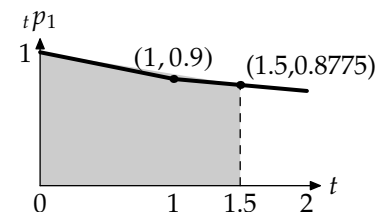
By linear interpolation, ${}_{1.5}p_1 = (0.5)(0.9 + 0.855) = 0.8775$.

The algebraic method splits the students into three groups: first year dropouts, second year (up to time 1.5) dropouts, and survivors. In each dropout group survival on the average is to the midpoint (0.5 years for the first group, 1.25 years for the second group) and survivors survive 1.5 years. Therefore

$$\dot{e}_{1:\overline{1.5}|} = 0.10(0.5) + (0.90 - 0.8775)(1.25) + 0.8775(1.5) = \boxed{1.394375} \quad (\text{D})$$

Alternatively, we could sum the two trapezoids making up the shaded area at the right.

$$\begin{aligned} \dot{e}_{1:\overline{1.5}|} &= (1)(0.5)(1 + 0.9) + (0.5)(0.5)(0.90 + 0.8775) \\ &= 0.95 + 0.444375 = \boxed{1.394375} \quad (\text{D}) \end{aligned}$$



8.17. Those who die survive 0.25 years on the average and survivors survive 0.5 years, so we have

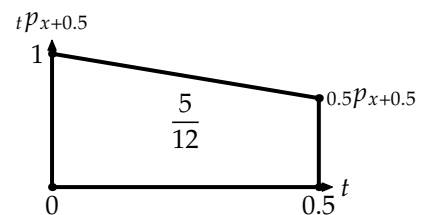
$$\begin{aligned} 0.25 {}_{0.5}q_{x+0.5} + 0.5 {}_{0.5}p_{x+0.5} &= \frac{5}{12} \\ 0.25 \left(\frac{0.5q_x}{1 - 0.5q_x} \right) + 0.5 \left(\frac{1 - q_x}{1 - 0.5q_x} \right) &= \frac{5}{12} \\ 0.125q_x + 0.5 - 0.5q_x &= \frac{5}{12} - \frac{5}{24}q_x \\ \frac{1}{2} - \frac{5}{12} &= \left(-\frac{5}{24} + \frac{1}{2} - \frac{1}{8} \right) q_x \\ \frac{1}{12} &= \frac{q_x}{6} \\ q_x &= \boxed{\frac{1}{2}} \end{aligned}$$

Alternatively, complete life expectancy is the area of the trapezoid shown on the right, so

$$\frac{5}{12} = 0.5(0.5)(1 + {}_{0.5}p_{x+0.5})$$

Then ${}_{0.5}p_{x+0.5} = \frac{2}{3}$, from which it follows

$$\begin{aligned} \frac{2}{3} &= \frac{1 - q_x}{1 - \frac{1}{2}q_x} \\ q_x &= \boxed{\frac{1}{2}} \end{aligned}$$



8.18. Survivors live 0.4 years and those who die live 0.2 years on the average, so

$$0.396 = 0.4 {}_{0.4}p_{55.2} + 0.2 {}_{0.4}q_{55.2}$$

Using the formula ${}_{0.4}q_{55.2} = 0.4q_{55}/(1 - 0.2q_{55})$ (equation (8.3)), we have

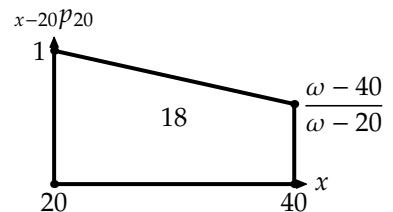
$$\begin{aligned} 0.4 \left(\frac{1 - 0.6q_{55}}{1 - 0.2q_{55}} \right) + 0.2 \left(\frac{0.4q_{55}}{1 - 0.2q_{55}} \right) &= 0.396 \\ 0.4 - 0.24q_{55} + 0.08q_{55} &= 0.396 - 0.0792q_{55} \\ 0.0808q_{55} &= 0.004 \\ q_{55} &= \frac{0.004}{0.0808} = 0.0495 \\ \mu_{55.2} &= \frac{q_{55}}{1 - 0.2q_{55}} = \frac{0.0495}{1 - 0.2(0.0495)} = \boxed{0.05} \end{aligned}$$

8.19. Since d_x is constant for all x and deaths are uniformly distributed within each year of age, mortality is uniform globally. We back out ω using equation (6.12), $\dot{e}_{x:\overline{n}|} = {}_n p_x(n) + nq_x(n/2)$:

$$\begin{aligned} 10 {}_{20}q_{20} + 20 {}_{20}p_{20} &= 18 \\ 10 \left(\frac{20}{\omega - 20} \right) + 20 \left(\frac{\omega - 40}{\omega - 20} \right) &= 18 \\ 200 + 20\omega - 800 &= 18\omega - 360 \\ 2\omega &= 240 \\ \omega &= 120 \end{aligned}$$

Alternatively, we can back out ω using the trapezoidal rule. Complete life expectancy is the area of the trapezoid shown to the right.

$$\begin{aligned} \dot{e}_{20:\overline{20}|} = 18 &= (20)(0.5) \left(1 + \frac{\omega - 40}{\omega - 20} \right) \\ 0.8 &= \frac{\omega - 40}{\omega - 20} \\ 0.8\omega - 16 &= \omega - 40 \\ 0.2\omega &= 24 \\ \omega &= 120 \end{aligned}$$



Once we have ω , we compute

$${}_{30|10}q_{30} = \frac{10}{\omega - 30} = \frac{10}{90} = \boxed{0.1111} \quad (\text{A})$$

8.20. We use equation (8.5) to obtain

$$\begin{aligned} 0.5 &= \frac{q_x}{1 - 0.5q_x} \\ q_x &= 0.4 \end{aligned}$$

Then $\dot{e}_{45:\overline{1}|} = 0.5(1 + (1 - 0.4)) = \boxed{0.8}$. (E)

8.21. According to the Standard Ultimate Life Table, $l_{30} = 99,727.30$, so we are looking for the age x such that $l_x = 0.75(99,727.30) = 74,795.475$. This is between ages 80 and 81. Using linear interpolation, since $l_{80} = 75,657.20$ and $l_{81} = 73,186.30$, we have

$$x = 80 + \frac{74,795.475}{75,657.20 - 73,186.30} = 80.3487$$

This is 50.3487 years into the future. $\frac{3}{4}$ of the people collect 50,000. We need $50,000 \left(\frac{3}{4} \right) \left(\frac{1}{1.12^{50.3487}} \right) = \boxed{124.73}$ per person.

8.22. Under constant force, ${}_s p_{x+t} = p_x^s$, so $p_x = {}_{1/4} p_{x+1/4}^4 = 0.98^4 = 0.922368$ and $q_x = 1 - 0.922368 = 0.077632$. Under uniform distribution of deaths,

$$\begin{aligned} {}_{1/4} p_{x+1/4} &= 1 - \frac{(1/4)q_x}{1 - (1/4)q_x} \\ &= 1 - \frac{(1/4)(0.077632)}{1 - (1/4)(0.077632)} \\ &= 1 - 0.019792 = \boxed{0.980208} \quad (\text{D}) \end{aligned}$$

8.23. Under constant force, ${}_s p_{x+t} = p_x^s$, so $p_x = 0.95^4 = 0.814506$, $q_x = 1 - 0.814506 = 0.185494$. Then under a uniform assumption,

$${}_{0.25} q_{x+0.1} = \frac{0.25q_x}{1 - 0.1q_x} = \frac{(0.25)(0.185494)}{1 - 0.1(0.185494)} = \boxed{0.047250} \quad (\text{B})$$

8.24. Using constant force, μ^A is a constant equal to $-\ln p_x = -\ln 0.75 = 0.287682$. Then

$$\begin{aligned} \mu_{x+s}^B &= \frac{q_x}{1 - s q_x} = 0.287682 \\ \frac{0.25}{1 - 0.25s} &= 0.287682 \\ 0.2877 - 0.25(0.287682)s &= 0.25 \\ s &= \frac{0.287682 - 0.25}{(0.25)(0.287682)} = \boxed{0.5239} \quad (\text{D}) \end{aligned}$$

8.25. We integrate ${}_t p_x$ from 0 to 2. Between 0 and 1, ${}_t p_x = e^{-t\mu_x}$.

$$\int_0^1 e^{-t\mu_x} dt = \frac{1 - e^{-\mu_x}}{\mu_x} = 0.99$$

Between 1 and 2, ${}_t p_x = p_x {}_{t-1} p_{x+1} = 0.98e^{-(t-1)\mu_{x+1}}$.

$$\int_1^2 e^{-(t-1)\mu_{x+1}} dt = \frac{1 - e^{-\mu_{x+1}}}{\mu_{x+1}} = 0.98$$

So the answer is $0.99 + 0.98(0.98) = \boxed{1.9504}$. (C)

8.26.

$$\begin{aligned} \ddot{e}_{80.4:\overline{0.8}|} &= \ddot{e}_{80.4:\overline{0.6}|} + 0.6p_{80.4} \ddot{e}_{81:\overline{0.2}|} \\ &= \frac{\int_{0.4}^1 0.9^t dt}{0.9^{0.4}} + 0.9^{0.6} \int_0^{0.2} 0.8^t dt \\ &= \frac{0.9^{0.6} - 1}{\ln 0.9} + (0.9^{0.6}) \frac{0.8^{0.2} - 1}{\ln 0.8} \\ &= 0.581429 + (0.938740)(0.195603) = \boxed{0.765049} \end{aligned}$$

8.27. Under uniform distribution, the numbers of deaths in each half of the year are equal, so if 120 deaths occurred in the first half of $x = 96$, then 120 occurred in the second half, and $l_{97} = 480 - 120 = 360$. Then if $0.5q_{97} = (360 - 288)/360 = 0.2$, then $q_{97} = 2 \cdot 0.5q_{97} = 0.4$, so $p_{97} = 0.6$. Under constant force, ${}_{1/2}p_{97} = p_{97}^{0.5} = \sqrt{0.6}$. The answer is $360\sqrt{0.6} = \boxed{278.8548}$. (D)

8.28. Let μ be the force of mortality in year 1. Then 10% survivorship means

$$e^{-\mu-3\mu} = 0.1$$

$$e^{-4\mu} = 0.1$$

The probability of survival 21 months given survival 3 months is the probability of survival 9 months after month 3, or $e^{-(3/4)\mu}$, times the probability of survival another 9 months given survival 1 year, or $e^{-(3/4)3\mu}$, which multiplies to $e^{-3\mu} = (e^{-4\mu})^{3/4} = 0.1^{3/4} = 0.177828$, so the death probability is $1 - 0.177828 = \boxed{0.822172}$. (E)

8.29. The exact value is:

$$\begin{aligned} F = {}_{10.5}p_0 &= \exp\left(-\int_0^{10.5} \mu_x dx\right) \\ \int_0^{10.5} (80-x)^{-0.5} dx &= -2(80-x)^{0.5}\Big|_0^{10.5} \\ &= -2(69.5^{0.5} - 80^{0.5}) = 1.215212 \\ {}_{10.5}p_0 &= e^{-1.215212} = 0.296647 \end{aligned}$$

To calculate $S_0(10.5)$ with constant force interpolation between 10 and 11, we have ${}_{0.5}p_{10} = p_{10}^{0.5}$, and ${}_{10.5}p_0 = {}_{10}p_0 \cdot {}_{0.5}p_{10}$, so

$$\begin{aligned} \int_0^{10} (80-x)^{-0.5} dx &= -2(70^{0.5} - 80^{0.5}) = 1.155343 \\ \int_{10}^{11} (80-x)^{-0.5} dx &= -2(69^{0.5} - 70^{0.5}) = 0.119953 \\ G = {}_{10.5}p_0 &= e^{-1.155343 - 0.5(0.119953)} = 0.296615 \end{aligned}$$

Then $F - G = 0.296647 - 0.296615 = \boxed{0.000032}$. (D)

Quiz Solutions

8-1. Notice that $\mu_{50.4} = \frac{q_{50}}{1-0.4q_{50}}$ while ${}_{0.6}q_{50.4} = \frac{0.6q_{50}}{1-0.4q_{50}}$, so ${}_{0.6}q_{50.4} = 0.6(0.01) = \boxed{0.006}$

8-2. The algebraic method goes: those who die will survive 0.3 on the average, and those who survive will survive 0.6.

$$\begin{aligned} {}_{0.6}q_{x+0.4} &= \frac{0.6(0.1)}{1-0.4(0.1)} = \frac{6}{96} \\ {}_{0.6}p_{x+0.4} &= 1 - \frac{6}{96} = \frac{90}{96} \\ \dot{e}_{x+0.4:\overline{0.6}|} &= \frac{6}{96}(0.3) + \frac{90}{96}(0.6) = \frac{55.8}{96} = \boxed{0.58125} \end{aligned}$$

The geometric method goes: we need the area of a trapezoid having height 1 at $x + 0.4$ and height $90/96$ at $x + 1$, where $90/96$ is ${}_{0.6}p_{x+0.4}$, as calculated above. The width of the trapezoid is 0.6. The answer is therefore $0.5(1 + 90/96)(0.6) = \boxed{0.58125}$.

8-3. Batteries failing in month 1 survive an average of 0.5 month, those failing in month 2 survive an average of 1.5 months, and those failing in month 3 survive an average of 2.125 months (the average of 2 and 2.25). By linear interpolation, ${}_{2.25}q_0 = 0.25(0.6) + 0.75(0.2) = 0.3$. So we have

$$\begin{aligned} \dot{e}_{\overline{0.225}|} &= q_0(0.5) + {}_1|q_0(1.5) + {}_2|_{0.25}q_0(2.125) + {}_{2.25}p_0(2.25) \\ &= (0.05)(0.5) + (0.20 - 0.05)(1.5) + (0.3 - 0.2)(2.125) + 0.70(2.25) = \boxed{2.0375} \end{aligned}$$

Practice Exam 1

SECTION A — Multiple-Choice

1. A life age 60 is subject to Gompertz's law with $B = 0.001$ and $c = 1.05$. Calculate $e_{60:\overline{2}|}$ for this life.

(A) 1.923 (B) 1.928 (C) 1.933 (D) 1.938 (E) 1.943

2. Your company sells whole life insurance policies. At a meeting with the Enterprise Risk Management Committee, it was agreed that you would limit the face amount of the policies sold so that the probability that the present value of the benefit at issue is greater than 1,000,000 is never more than 0.05.

You are given:

- (i) The insurance policies pay a benefit equal to the face amount b at the moment of death.
- (ii) The force of mortality is $\mu_x = 0.001(1.05^x)$, $x > 0$
- (iii) $\delta = 0.06$

Determine the largest face amount b for a policy sold to a purchaser who is age 45.

(A) 1,350,000 (B) 1,400,000 (C) 1,450,000 (D) 1,500,000 (E) 1,550,000

3. For an annual premium 2-year term insurance on (60) with benefit b payable at the end of the year of death, you are given

(i)

t	p_{60+t-1}
1	0.98
2	0.96

- (ii) The annual net premium is 25.41.
- (iii) $i = 0.05$.

Determine the revised annual net premium if an interest rate of $i = 0.04$ is used.

(A) 25.59 (B) 25.65 (C) 25.70 (D) 25.75 (E) 25.81

4. In a three-state Markov chain, you are given the following forces of transition:

$$\mu_t^{01} = 0.05 \qquad \mu_t^{10} = 0.04 \qquad \mu_t^{02} = 0.03 \qquad \mu_t^{12} = 0.10$$

All other forces of transition are 0.

Calculate the probability of an entity in state 0 at time 0 transitioning to state 1 before time 5 and staying there until time 5, then transitioning to state 0 before time 10 and staying there until time 10.

(A) 0.017 (B) 0.018 (C) 0.019 (D) 0.020 (E) 0.021

5. A study is performed on number of days required to underwrite a policy. The results of the study are:

Number of Days	Number of Policies
(0,10]	11
(10,20]	x
(20,50]	y

An ogive is used to interpolate between interval boundaries.

You are given:

- (i) $\hat{F}(15) = 0.35$
(ii) $\hat{f}(15) = 1/30$

Determine x .

- (A) 16 (B) 18 (C) 20 (D) 22 (E) 24

6. You are given the following profit test for a 10-year term insurance of 100,000 on (x) :

t	${}_{t-1}V$	P	E_t	I_t	bq_{x+t-1}	$p_{x+t-1} {}_tV$
0			-350			
1	0	1000	0	60.0	500	447.75
2	450	1000	20	85.8	600	795.20
3	800	1000	20	106.8	700	1092.30
4	1100	1000	20	124.8	800	1289.60
5	1300	1000	20	136.8	900	1412.18
6	1425	1000	20	144.3	1000	1435.50
7	1450	1000	20	145.8	1100	1285.70
8	1300	1000	20	136.8	1200	1037.40
9	1050	1000	20	121.8	1300	641.55
10	650	1000	20	97.8	1400	0.00

Which of the following statements is true?

- I. The interest rate used in the calculation is $i = 0.06$.
II. At time 5, the reserve per survivor is 1425.
III. The profit signature component for year 3 is 92.81
- (A) I and II only (B) I and III only (C) II and III only (D) I, II, and III
(E) The correct answer is not given by (A), (B), (C), or (D).

7. For a fully continuous whole life insurance of 1000 on (x) :

- (i) The gross premium is paid at an annual rate of 25.
(ii) The variance of future loss is 2,000,000.
(iii) $\delta = 0.06$

Employees are able to obtain this insurance for a 20% discount.

Determine the variance of future loss for insurance sold to employees.

- (A) 1,281,533 (B) 1,295,044 (C) 1,771,626 (D) 1,777,778 (E) 1,825,013

8. In a mortality study, the cumulative hazard function is estimated using the Nelson-Åalen estimator. There are initially 41 lives. There are no censored observations before the first time of deaths, y_1 .

The number of deaths at time y_1 is less than 6.

Using Klein's variance formula, $\widehat{\text{Var}}(\hat{H}(y_1)) = 0.000580$.

Determine the number of deaths at time y_1 .

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. For two lives (50) and (60) with independent future lifetimes:

(i) $\mu_{50+t} = 0.002t, \quad t > 0$

(ii) $\mu_{60+t} = 0.003t, \quad t > 0$

Calculate ${}_{20}q_{50:60}^1 - {}_{20}q_{50:60}^2$.

- (A) 0.17 (B) 0.18 (C) 0.30 (D) 0.31 (E) 0.37

10. For a fully discrete 20-year deferred whole life insurance of 1000 on (50), you are given:

(i) Premiums are payable for 20 years.

(ii) The net premium is 12.

(iii) Deaths are uniformly distributed between integral ages.

(iv) $i = 0.1$

(v) ${}_9V = 240$ and ${}_{9.5}V = 266.70$.

Calculate ${}_{10}V$, the net premium reserve at the end of year 10.

- (A) 272.75 (B) 280.00 (C) 281.40 (D) 282.28 (E) 282.86

11. In a study of 10 lives, you are given the following data:

Life	d_i	x_i	u_i
1	2.0	3.1	
2	2.5	4.0	
3	3.0		3.2
4	3.4		4.0
5	3.8	6.2	
6	4.0		5.2
7	4.0	8.4	
8	4.0		5.2
9	4.2	5.2	
10	4.4		8.4

Calculate the Nelson-Åalen estimate of $S(7 | X > 2)$.

- (A) 0.23 (B) 0.25 (C) 0.27 (D) 0.9 (E) 0.31

12. A life age 90 is subject to mortality following Makeham's law with $A = 0.0005$, $B = 0.0008$, and $c = 1.07$.

Curtate life expectancy for this life is 6.647 years.

Using Woolhouse's formula with three terms, compute complete life expectancy for this life.

- (A) 7.118 (B) 7.133 (C) 7.147 (D) 7.161 (E) 7.176

13. You are given that $\mu_x = 0.002x + 0.005$.

Calculate ${}_5|q_{20}$.

- (A) 0.015 (B) 0.026 (C) 0.034 (D) 0.042 (E) 0.050

14. For a temporary life annuity-due of 1 per year on (30), you are given:

- (i) The annuity makes 20 certain payments.
- (ii) The annuity will not make more than 40 payments.
- (iii) Mortality follows the Standard Ultimate Life Table.
- (iv) $i = 0.05$

Determine the expected present value of the annuity.

- (A) 17.79 (B) 17.83 (C) 17.87 (D) 17.91 (E) 17.95

15. For a fully discrete whole life insurance on (35) with face amount 100,000, you are given the following assumptions and experience for the fifth year:

	Assumptions	Actual
q_{39}	0.005	0.006
Surrender probability	0.05	0.06
Annual expenses	20	30
Settlement expenses—death	100	80
Settlement expenses—surrender	50	40
i	0.05	0.045

You are also given:

- (i) The gross premium is 1725.
- (ii) Reserves are gross premium reserves.
- (iii) The gross premium reserve at the end of year 4 is 6000.
- (iv) The cash surrender value for the fifth year is 6830.
- (v) The surrender probability is based on the multiple-decrement table.

The fifth year gain is analyzed in the order of interest, surrender, death, expense.

Determine the fifth year surrender gain.

- (A) -7.9 (B) -7.7 (C) -7.5 (D) 7.7 (E) 7.9

16. In a double-decrement model, with decrements (1) and (2), you are given, for all $t > 0$:

- (i) ${}_t p_x^{(1)} = 10/(10 + t)$
- (ii) ${}_t p_x^{(2)} = (10/(10 + t))^3$

Determine $q_x^{(1)}$.

- (A) 0.068 (B) 0.074 (C) 0.079 (D) 0.083 (E) 0.091

17. Mortality in 2015 follows the Standard Ultimate Life Table. Mortality improvement factors are

x	$\phi(x)$
60	0.032
61	0.030
62	0.027
63	0.024
64	0.020

Determine the first year for which q_{60} is less than 0.002.

- (A) 2031 (B) 2032 (C) 2033 (D) 2034 (E) 2035

18. For an insurance with face amount 100,000, you are given:

(i)

$$\frac{d}{dt} {}_tV = 100$$

(ii) $P = 1380$

(iii) $\delta = 0.05$

(iv) $\mu_{x+t} = 0.03$

Determine ${}_tV$.

- (A) 21,000 (B) 21,500 (C) 22,000 (D) 22,500 (E) 23,000

19. In a mortality study on 5 lives, you are given the following information:

Entry age	Exit age	Cause of exit
62.3	65.1	End of study
63.5	66.0	Withdrawal
64.0	65.7	Withdrawal
64.2	65.5	Death
64.7	67.7	End of study

Calculate the absolute difference between the actuarial estimate and the exact exposure estimate of q_{65} .

- (A) 0.002 (B) 0.006 (C) 0.010 (D) 0.014 (E) 0.018

20. For a defined benefit pension plan, you are given

(i) Accrual rate is 1.6%

(ii) The pension benefit is a monthly annuity-due payable starting at age 65, based on final salary.

(iii) No benefits are payable for death in service.

(iv) There are no exits other than death before retirement.

(v) Salaries increase 3% per year.

(vi) $i = 0.04$

An employee enters the plan at age 32. At age 45, the accrued liability for the pension, using the projected unit credit method, is 324,645.

Calculate the normal contribution for this employee for the year beginning at age 45.

- (A) 24,000 (B) 25,000 (C) 26,000 (D) 27,000 (E) 28,000

SECTION B — Written-Answer

1. (11 points) A special 5-year term insurance on (55) pays 1000 plus the net premium reserve at the end of the year of death. A single premium is paid at inception. You are given:

- (i) Mortality follows the Standard Ultimate Life Table.
- (ii) $i = 0.05$

- (a) (2 points) Calculate the net single premium for this policy.
- (b) (3 points) Using the recursive formula for reserves, calculate net premium reserves for the policy at times 1, 2, 3, and 4.
- (c) (2 points) Suppose the policy, in addition to paying death benefits, pays the single premium at the end of 5 years to those who survive.
Calculate the revised single premium.
- (d) (2 points) Calculate the net single premium for an otherwise similar policy that pays 1000, but not the net premium reserve, at the end of the year of death.
- (e) (2 points) Calculate the net single premium for an otherwise similar policy that pays 1000 plus the net single premium, but not the net premium reserve, at the end of the year of death.

2. (9 points) A one-year term life insurance on (x) pays 2000 at the moment of decrement 1 and 1000 at the moment of decrement 2. You are given

- (i) $q_x^{(1)} = 0.1$
- (ii) $q_x^{(2)} = 0.3$
- (iii) $\delta = 0.04$

- (a) (3 points) The decrements are uniform in the multiple decrement table.
Calculate the EPV of the insurance.
- (b) (3 points) The decrements are uniform in the associated single decrement tables.
Calculate the EPV of the insurance.
- (c) (3 points) The forces of decrement are constant.
Calculate the EPV of the insurance.

3. (8 points) A continuous whole life annuity on (60) pays 100 per year.

You are given:

- (i) Mortality follows $l_x = 1000(100 - x)$, $0 \leq x \leq 100$.
- (ii) $\delta = 0.05$.

- (a) (2 points) Calculate the probability that the present value of payments on the annuity is greater than its net single premium.
Use the following information for (b) and (c):
In addition to the annuity payments, a death benefit of 1000 is paid at the moment of death if death occurs within the first ten years.
- (b) (4 points) Calculate the probability that the present value of payments on the annuity (including the death benefit) is greater than its net single premium.
- (c) (2 points) Calculate the minimum value of the present value of payments.

4. (10 points) A special whole life insurance on (35) pays a benefit at the moment of death. You are given:

- (i) The benefit for death in year k is $9000 + 1000k$, but never more than 20,000.
- (ii) Mortality follows the Standard Ultimate Life Table.
- (iii) $i = 0.05$.
- (iv) $1000(IA)_{35:\overline{10}|}^1 = 22.28$
- (v) Premiums are payable monthly.

- (a) (2 points) Calculate the net single premium for the policy assuming uniform distribution of deaths between integral ages.
- (b) (2 points) Calculate the net single premium for a whole life annuity-due annuity on (35) of 1 per month using Woolhouse's formula and approximating $\mu_x = -0.5(\ln p_{x-1} + \ln p_x)$.
- (c) (1 point) Calculate the net premium payable monthly, using the assumptions and methods of parts (a) and (b).
- (d) (3 points) Calculate the net premium reserve at time 10, using the same method as was used to calculate the net premium.

Suppose that instead of the benefit pattern of (i), the death benefit of the insurance is $11,000 - 1000k$, but never less than 1000.

- (e) (2 points) Calculate the net single premium for the insurance, assuming uniform distribution of deaths between integral ages.

5. (7 points) Your company conducts a mortality study based on policy data from Jan. 1, 2015 through Dec. 31, 2016. The data for estimating q_{40} includes 380 policies with policyholders who were younger than age 40 on Jan. 1, 2015 and older than age 40 on Dec. 31, 2016, and who neither died nor withdrew during the two-year period. In addition, the data includes the following six policies:

Birth date	Policy issue date	Withdrawal date	Death date
Apr. 1, 1974	Feb. 1, 2015	—	—
June 1, 1974	Feb. 1, 2014	—	Feb. 1, 2015
Sept. 1, 1974	June 1, 2014	Aug. 1, 2015	—
Jan. 1, 1975	Jan. 1, 2008	—	May 1, 2015
Mar. 1, 1975	Mar. 1, 2011	Dec. 1, 2016	—
May 1, 1975	Dec. 1, 2005	Oct. 1, 2015	—

- (a) (2 points) You use the actuarial estimator to estimate q_{40} . Calculate the estimate.
- (b) (2 points) Your boss suggests that using the Kaplan-Meier estimator would be more precise. Calculate the Kaplan-Meier estimate of q_{40} .
- (c) (2 points) Estimate the standard deviation of the Kaplan-Meier estimate of q_{40} using Greenwood's formula.
- (d) (1 point) Give three reasons that life insurance companies use approximations such as the actuarial estimator to estimate mortality rates, rather than other estimation methods.

6. (11 points) The ZYX Company offers a defined benefit pension plan with the following provisions:

- At retirement at age 65, the plan pays a monthly whole life annuity-due providing annual income that accrues at the rate of 1.5% of final salary up to 100,000 and 2% of the excess of final salary over 100,000 for each year of service.
- There is no early retirement.
- There are no other benefits.

The following assumptions are made:

- (i) No employees exit the plan before retirement except by death.
- (ii) Retirement occurs at the beginning of each year.
- (iii) Pre-retirement mortality follows the Standard Ultimate Life Table.
- (iv) Salaries increase 3% each year.
- (v) $i = 0.05$.
- (vi) $\ddot{a}_{65}^{(12)} = 11$.

The ZYX Company has the following 3 employees on January 1, 2020:

Name	Exact Age	Years of Service	Salary in Previous Year
Cramer	55	20	120,000
Liu	35	5	50,000
Smith	50	10	100,000

- (a) (3 points) Show that the actuarial liability using TUC is 333,000 to the nearest 1000. You should answer to the nearest 10.
- (b) (3 points) Calculate the normal contribution for 2020 using TUC.
- (c) (1 point) Calculate the replacement ratio for Cramer if he retires at age 65 and the salary increases follow assumptions.
- (d) (2 points) Fifteen years later, Smith retires. Smith's salary increases have followed assumptions. Smith would prefer an annual whole life annuity-due.
Calculate the annual payment that is equivalent to the pension plan's monthly benefit using Woolhouse's formula to two terms.
- (e) (2 points) On January 2, 2020, a pension consultant suggests that $q_{39} = 0.00035$ is a better estimate of mortality than the rate in the Standard Ultimate Life Table. No other mortality rate changes are suggested.
Recalculate the actuarial liability under TUC as of January 1, 2020 using this new assumption.

Solutions to the above questions begin on page 1583.

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	E	6	A	11	E	16	C
2	A	7	C	12	A	17	B
3	C	8	A	13	D	18	B
4	A	9	B	14	C	19	A
5	C	10	D	15	E	20	B

Practice Exam 1

SECTION A — Multiple-Choice

1. [Section 6.2] By formula (5.2),

$$p_{60} = \exp\left(-0.001(1.05^{60})\left(\frac{0.05}{\ln 1.05}\right)\right) = 0.981040$$
$${}_2p_{60} = \exp\left(-0.001(1.05^{60})\left(\frac{1.05^2 - 1}{\ln 1.05}\right)\right) = 0.961518$$

Then $e_{60:\overline{2}|} = 0.981040 + 0.961518 = \boxed{1.9426}$. (E)

2. [Lesson 15] The present value of the benefit decreases with increasing survival time, so the 95th percentile of the present value of the insurance corresponds to the 5th percentile of survival time. The survival probability is

$${}_t p_{45} = \exp\left(-\int_0^t 0.001(1.05^{45+u})du\right)$$
$$-\ln {}_t p_{45} = \frac{0.001(1.05^{45+u})}{\ln 1.05}\bigg|_0^t$$
$$= \frac{0.001(1.05^{45+t} - 1.05^{45})}{\ln 1.05}$$

Setting ${}_t p_{45} = 0.95$,

$$\frac{0.001(1.05^{45+t} - 1.05^{45})}{\ln 1.05} = -\ln 0.95$$
$$1.05^{45+t} = (-1000 \ln 0.95)(\ln 1.05) + 1.05^{45} = 11.48762$$
$$1.05^t = \frac{11.48762}{1.05^{45}} = 1.27853$$
$$t = \frac{\ln 1.27853}{\ln 1.05} = 5.0361$$

The value of Z if death occurs at $t = 5.0361$ is $b e^{-5.0361(0.06)}$, so the largest face amount is $1,000,000 e^{5.0361(0.06)} = \boxed{1,352,786}$. (A)

3. [Lesson 26] The revised premium for the entire policy is 25.41 times the ratio of the revised premium per unit at 4% to the original premium per unit at 5%.

We calculate the original net premium per unit, $P_{60:\overline{2}|}^1$.

$$\begin{aligned}\ddot{a}_{60:\overline{2}|} &= 1 + \frac{0.98}{1.05} = 1.93333 \\ A_{60:\overline{2}|}^1 &= \frac{0.02}{1.05} + \frac{(0.98)(0.04)}{1.05^2} = 0.054603 \\ P_{60:\overline{2}|}^1 &= \frac{A_{60:\overline{2}|}^1}{\ddot{a}_{60:\overline{2}|}} = \frac{0.054603}{1.93333} = 0.028243\end{aligned}$$

Now we recalculate at 4%. Call the revised premium $P_{60:\overline{2}|}^1$.

$$\begin{aligned}\ddot{a}_{60:\overline{2}|} &= 1 + \frac{0.98}{1.04} = 1.94231 \\ A_{60:\overline{2}|}^1 &= \frac{0.02}{1.04} + \frac{(0.98)(0.04)}{1.04^2} = 0.055473 \\ P_{60:\overline{2}|}^1 &= \frac{0.055473}{1.94231} = 0.028561\end{aligned}$$

So the revised premium for benefit b is $25.41(0.028561/0.028243) = \boxed{25.696}$. (C)

4. [Section 45.1] Let ${}_5p_0^{\overline{01}}$ be the probability that an entity in state 0 at time 0 transitions to state 1 before time 5 and stays there until time 5, and let ${}_5p_5^{\overline{10}}$ be the probability that an entity in state 1 at time 5 transitions to state 0 before time 10 and stays there until time 10. We'll use formula (45.9) for both transitions. Notice that the formula is the same with 0 and 1 switched, except that ${}_5p_0^{\overline{01}}$ uses $\mu^{01} = 0.05$ and ${}_5p_5^{\overline{10}}$ uses $\mu^{10} = 0.04$ outside the parentheses.

$$\begin{aligned}\frac{e^{-\mu^{0\bullet}t}}{\mu^{1\bullet} - \mu^{0\bullet}} + \frac{e^{-\mu^{1\bullet}t}}{\mu^{0\bullet} - \mu^{1\bullet}} &= \frac{e^{-0.08(5)}}{0.14 - 0.08} + \frac{e^{-0.14(5)}}{0.08 - 0.14} = 2.89558 \\ {}_5p_0^{\overline{01}} &= 0.05(2.89558) = 0.14478 \\ {}_5p_5^{\overline{10}} &= 0.04(2.89558) = 0.11582\end{aligned}$$

The answer is $(0.14478)(0.11582) = \boxed{0.01677}$. (A)

5. [Section 64.2]

$$\begin{aligned}\hat{F}(15) &= \frac{11 + 0.5x}{11 + x + y} = 0.35 \\ \hat{f}(15) &= \frac{x}{10(11 + x + y)} = 1/30\end{aligned}$$

Multiply the second equation by 10.

$$\hat{f}(15) = \frac{x}{11 + x + y} = 1/3$$

Divide it into the $\hat{F}(15)$ equation.

$$\begin{aligned}\frac{11 + 0.5x}{x} &= 1.05 \\ 11 + 0.5x &= 1.05x \\ 11 &= 0.55x \\ x &= \boxed{20} \quad \text{(C)}\end{aligned}$$

6. [Lesson 73]

- I From the row for year 1, with 0 reserves and expenses, we see that $I_t/P_t = 0.06$, so the interest rate is 0.06. ✓
- II Looking at the line for $t = 6$, we see that the reserve per survivor to time $t - 1 = 5$ is 1425. ✓
- III First, the profit in year 3 is $800 + 1000 - 20 + 106.8 - 700 - 1092.3 = 94.50$. We deduce survivorship from the bq_{x+t-1} column, and we see that the mortality rates in the first two years are 0.005 and 0.006, so the profit signature component of year 3 is $(0.995)(0.994)(94.50) = 93.46$. ✗

(A)

7. [Lesson 32] The variance of future loss for a gross premium of 25 is

$$\begin{aligned} 2,000,000 &= \text{Var} \left(v^{T_x} \left(1000 + \frac{25}{0.06} \right) \right)^2 \\ &= \text{Var} \left(v^{T_x} \right) (2,006,944) \end{aligned}$$

If we replace 25 with 20 (for a 20% discount) in the above formula, it becomes

$$\begin{aligned} \text{Var}({}_0L) &= \text{Var} \left(v^{T_x} \left(1000 + \frac{20}{0.06} \right) \right)^2 \\ &= \text{Var} \left(v^{T_x} \right) (1,777,778) \end{aligned}$$

We see that this is $1,777,778/2,006,944$ times the given variance, so the final answer is

$$\text{Var}({}_0L) = \frac{1,777,778}{2,006,944} (2,000,000) = \boxed{1,771,626} \quad (\text{C})$$

8. [Section 68.1]

$$\frac{s_1(41 - s_1)}{41^3} = 0.00058$$

$$s_1(41 - s_1) = 40$$

$$s_1 = \boxed{1}, 40 \quad (\text{A})$$

9. [Lesson 58]
- ${}_{20}q_{50:\overline{60}} - {}_{20}q_{50:\overline{60}}^2 = {}_{20}q_{50} {}_{20}p_{60}$
- , and

$$\begin{aligned} {}_{20}q_{50} &= 1 - \exp \left(- \int_0^{20} 0.002t \, dt \right) \\ &= 1 - e^{-0.001(20)^2} = 1 - 0.670320 = 0.329680 \end{aligned}$$

$$\begin{aligned} {}_{20}p_{60} &= \exp \left(- \int_0^{20} 0.003t \, dt \right) \\ &= e^{-0.0015(20)^2} = 0.548812 \end{aligned}$$

$${}_{20}q_{50} {}_{20}p_{60} = (0.329680)(0.548812) = \boxed{0.180932} \quad (\text{B})$$

10. [Section 42.2] We need to back out
- q_{59}
- . We use reserve recursion. Since the insurance is deferred,
- $1000q_{59}$
- is not subtracted from the left side.

$$\begin{aligned} ({}_9V + P)(1.1^{0.5}) &= {}_{9.5}V(1 - 0.5q_{59}) \\ 252(1.1^{0.5}) &= 266.70 - 133.35q_{59} \\ q_{59} &= \frac{2.40017}{133.35} = 0.018 \end{aligned}$$

Then the net premium reserve at time 10 is, by recursion from time 9,

$$\frac{252(1.1)}{1 - 0.018} = \boxed{282.28} \quad (\text{D})$$

11. [Lesson 66] Risk sets are:

At time 3.1, lives 1,2,3

At time 4.0, lives 2,4,5

At time 5.2, lives 5–10

At time 6.2, lives 5,7,10

y_i	r_i	s_i	$\hat{H}(7 X \geq 2)$
3.1	3	1	1/3
4.0	3	1	2/3
5.2	6	1	5/6
6.2	3	1	7/6

$$\hat{S}(7 | X > 2) = e^{-7/6} = \boxed{0.3114} \quad (\text{E})$$

12. [Section 24.2] By equation (24.10),

$$\ddot{e}_x = e_x + \frac{1}{2} - \frac{1}{12}\mu_x$$

Force of mortality for (90) is $\mu_{90} = 0.0005 + 0.0008(1.07^{90}) = 0.353382$. Thus

$$\ddot{e}_{90} = 6.647 + 0.5 - \frac{1}{12}(0.353382) = \boxed{7.118} \quad (\text{A})$$

13. [Lesson 4] ${}_5|q_{20} = (S_0(25) - S_0(26))/S_0(20)$, so we will calculate these three values of $S_0(x)$. (Equivalently, one could calculate ${}_5p_{20}$ and ${}_6p_{20}$ and take the difference.) The integral of μ_x is

$$\int_0^x \mu_u \, du = \left(\frac{0.002u^2}{2} + 0.005u \right) \Big|_0^x = 0.001x^2 + 0.005x$$

so

$$S_0(20) = \exp\left(- (0.001(20^2) + 0.005(20))\right) = \exp(-0.5) = 0.606531$$

$$S_0(25) = \exp\left(- (0.001(25^2) + 0.005(25))\right) = \exp(-0.75) = 0.472367$$

$$S_0(26) = \exp\left(- (0.001(26^2) + 0.005(26))\right) = \exp(-0.806) = 0.446641$$

and the answer is

$${}_5|q_{20} = \frac{0.472367 - 0.446641}{0.606531} = \boxed{0.042415} \quad (\text{D})$$

14. [Lesson 19] This annuity is the sum of a 20-year certain annuity-due and a 20-year deferred 20-year temporary life annuity due.

$$\ddot{a}_{20} = \frac{1 - (1/1.05)^{20}}{1 - 1/1.05} = 13.08532$$

$$\begin{aligned} {}_{20}|\ddot{a}_{30:\overline{20}|} &= {}_{20}E_{30} \ddot{a}_{50:\overline{20}|} \\ &= (0.37254)(12.8428) = 4.78446 \end{aligned}$$

The expected present value of the annuity is $13.08532 + 4.78446 = \boxed{17.8698}$. (C)

15. [Lesson 74] Surrender gain per surrender is the ending reserve (which is released into profit) minus the benefit paid and minus expenses. The ending gross premium reserve is

$${}_5V = \frac{(6000 + 1725 - 20)(1.05) - (100,000 + 100)(0.005) - (6830 + 50)(0.05)}{1 - 0.05 - 0.005} = 7667.46$$

Using assumed expenses, the surrender gain per surrender is $7667.46 - (6830 + 50) = 787.46$. The gain is $(0.06 - 0.05)(787.46) = \boxed{7.8746}$. (E)

16. [Lesson 50]

$$\begin{aligned} {}_t p_x^{(\tau)} &= \left(\frac{10}{10+t}\right)\left(\frac{10}{10+t}\right)^3 = \left(\frac{10}{10+t}\right)^4 \\ \mu_{x+t}^{(1)} &= -\frac{d \ln {}_t p_x^{(1)}}{dt} \\ &= -\frac{d(\ln 10 - \ln(10+t))}{dt} \\ &= \frac{1}{10+t} \\ q_x^{(1)} &= \int_0^1 {}_t p_x^{(\tau)} \mu_{x+t}^{(1)} dt \\ &= \int_0^1 \left(\frac{10}{10+t}\right)^4 \left(\frac{1}{10+t}\right) dt \\ &= \int_0^1 \frac{10^4 dt}{(10+t)^5} \\ &= -\left(\frac{10^4}{4}\right) \left(\frac{1}{(10+t)^4}\right) \Big|_0^1 \\ &= \left(\frac{10^4}{4}\right) \left(\frac{1}{10^4} - \frac{1}{11^4}\right) \\ &= \boxed{0.079247} \quad (\text{C}) \end{aligned}$$

17. [Section 10.1]

$$\begin{aligned} 0.003398(1 - 0.032)^x &< 0.002 \\ x \ln 0.968 &< \ln \frac{0.002}{0.003398} = \ln 0.588582 \\ x &> \frac{\ln 0.588582}{\ln 0.968} = 16.297 \end{aligned}$$

The first year is $2015 + 17 = \boxed{2032}$. (B)

18. [Section 42.3]

$$\begin{aligned} 100 &= (0.05 + 0.03) {}_t V + 1380 - 100,000(0.03) = 0.08 {}_t V - 1620 \\ {}_t V &= \frac{1720}{0.08} = \boxed{21,500} \quad (\text{B}) \end{aligned}$$

19. [Section 69.1] The exact exposure at age 65 is, $(65.1 - 65) + (66 - 65) + (65.7 - 65) + (65.5 - 65) + (66 - 65) = 3.3$. Then the exact exposure estimate is $\hat{q}_{65} = 1 - e^{-1/3.3} = 0.261423$.

The actuarial exposure for the death is 1 instead of 0.5, so actuarial exposure for the group is $3.3 + 0.5 = 3.8$. The actuarial estimate is $\hat{q}_{65} = 1/3.8 = 0.263158$.

The absolute difference is $0.263158 - 0.261423 = \boxed{0.0017}$. (A)

20. [Section 71.2] Using PUC, if there are no exit benefits and accruals are the same percentage each year, the normal contribution is the initial accrued liability divided by the number of years of service, or $324,645/13 = \boxed{24,973}$. (B)

SECTION B — Written-Answer

1. [Section 40.2]

(a) The reserve at time 5 is 0, so the single premium P is determined from

$$0 = P(1+i)^5 - 1000 \sum_{k=1}^5 q_{55+k-1}(1+i)^{5-k}$$

or

$$\begin{aligned} P &= 1000 \sum_{k=1}^5 q_{55+k-1} v^k \\ &= 1000 \left(\frac{0.001993}{1.05} + \frac{0.002212}{1.05^2} + \frac{0.002459}{1.05^3} + \frac{0.002736}{1.05^4} + \frac{0.003048}{1.05^5} \right) \\ &= \boxed{10.6677} \end{aligned}$$

(b) Because the net premium reserve is paid on death, the recursion does not divide by p_x .

$$\begin{aligned} 10.6677(1.05) - 1.993 &= 9.2081 \\ 9.2081(1.05) - 2.212 &= 7.4565 \\ 7.4565(1.05) - 2.459 &= 5.3703 \\ 2.9029(1.05) - 2.736 &= 2.9029 \end{aligned}$$

Although not required, you could check the calculation by doing one more recursion: $2.9029(1.05) - 3.048 = 0$.

(c) The reserve at time 5 is P , so the single premium P is determined from

$$P = P(1+i)^5 - 1000 \sum_{k=1}^5 q_{55+k-1}(1+i)^{5-k}$$

or

$$P(1-v^5) = 1000 \sum_{k=1}^5 q_{55+k-1} v^k$$

We divide the answer to part (a) by $1 - v^5$:

$$10.6677/(1 - 1/1.05^5) = \boxed{49.2795}$$

(d)

$$1000A_{55:\overline{5}|} = 1000(A_{55} - {}_5E_{55}A_{60}) = 235.24 - (0.77382)(290.28) = \boxed{10.6155}$$

(e)

$$P = (1000 + P)A_{55:\overline{5}|}$$

$$P = \frac{10.6155}{1 - 0.0106155} = \boxed{10.7294}$$

2. [Lessons 51 and 53]

(a)

$$p_x^{(\tau)} = (0.9)(0.7) = 0.63$$

$$q_x^{(1)} = (0.37) \left(\frac{\ln 0.9}{\ln 0.63} \right) = 0.084373$$

$$q_x^{(2)} = (0.37) \left(\frac{\ln 0.7}{\ln 0.63} \right) = 0.285627$$

Since the decrements are uniform in the multiple decrement table, ${}_s p_x^{(\tau)} \mu_{x+s}^{(j)}$ is constant and equal to $q_x^{(j)}$. The EPV of the insurance is

$$\int_0^1 v^s {}_s p_x^{(\tau)} (2000\mu_{x+s}^{(1)} + 1000\mu_{x+s}^{(2)}) ds = (2000(0.084373) + 1000(0.285627)) \left(\frac{1 - e^{-0.04}}{0.04} \right) = \boxed{445.41}$$

(b) The forces of mortality are $\mu_{x+s}^{(1)} = \frac{0.1}{1-0.1s}$ and $\mu_{x+s}^{(2)} = \frac{0.3}{1-0.3s}$. Also, ${}_s p_x^{(\tau)} = (1 - 0.1s)(1 - 0.3s)$. So the EPV of the insurance is

$$\begin{aligned} \text{EPV} &= \int_0^1 v^s (1 - 0.1s)(1 - 0.3s) \left(2000 \frac{0.1}{1 - 0.1s} + 1000 \frac{0.3}{1 - 0.3s} \right) ds \\ &= \int_0^1 e^{-0.04s} (500 - 90s) ds \\ &= -\frac{e^{-0.04s}}{0.04} (500 - 90s) \Big|_0^1 - 90 \int_0^1 \frac{e^{-0.04s}}{0.04} ds \\ &= \frac{500 - 410e^{-0.04}}{0.04} - \frac{90(1 - e^{-0.04})}{0.04^2} = \boxed{446.31} \end{aligned}$$

(c) The forces of decrement are $-\ln p_x^{(j)}$, or $\mu_x^{(1)} = -\ln 0.9$ and $\mu_x^{(2)} = -\ln 0.7$. The probability of survival from both decrements under constant force is

$${}_s p_x^{(\tau)} = {}_s p_x^{(1)} {}_s p_x^{(2)} = (0.9^s)(0.7^s) = 0.63^s$$

The EPV of the insurance is

$$\begin{aligned} \text{EPV} &= \int_0^1 v^s {}_s p_x^{(\tau)} (2000\mu_x^{(1)} + 1000\mu_x^{(2)}) ds \\ &= \int_0^1 e^{(-0.04 + \ln 0.63)s} \underbrace{(-2000 \ln 0.9 - 1000 \ln 0.7)}_{567.396} ds \\ &= 567.396 \int_0^1 e^{(-0.04 + \ln 0.63)s} ds \\ &= \frac{567.396}{-\ln 0.63 + 0.04} (1 - 0.63e^{-0.04}) = \boxed{446.09} \end{aligned}$$

3. [Lesson 22]

- (a) First let's calculate the net single premium. We can ignore the 100 per year factor; it just scales up the numbers.

$$\bar{A}_{60} = \frac{1 - e^{-0.05(40)}}{0.05(40)} = 0.432332$$

$$\bar{a}_{60} = \frac{1 - 0.432332}{0.05} = 11.35335$$

 $\bar{a}_{\overline{T}|} = \bar{a}_{60}$ when:

$$\frac{1 - e^{-0.05t}}{0.05} = \frac{1 - 0.432332}{0.05}$$

$$e^{-0.05t} = 0.432332$$

$$t = -\frac{\ln 0.432332}{0.05} = 16.77121$$

The probability that $T_{60} > 16.77121$ is $1 - 16.77121/40 = \boxed{0.58072}$.

- (b) First let's calculate the net single premium.

$$\bar{A}_{60:\overline{10}|}^1 = \frac{1 - e^{-0.05(10)}}{0.05(40)} = 0.196735$$

$$1000\bar{A}_{60:\overline{10}|}^1 + 100\bar{a}_{60} = 196.735 + 1135.335 = 1332.070$$

The present value of payments may be higher than 1332.070 in the first 10 years. However, let's begin by calculating the time $t > 10$ at which the present value of payments is higher than 1332.070.

$$100\left(\frac{1 - e^{-0.05t}}{0.05}\right) = 1332.070$$

$$e^{-0.05t} = 0.333965$$

$$t = -\frac{\ln 0.333965}{0.05} = 21.93438$$

Now let's determine the time $t < 10$ for which the present value of payments is 1332.070.

$$1000e^{-0.05t} + 100\left(\frac{1 - e^{-0.05t}}{0.05}\right) = 1332.07$$

$$-1000e^{-0.05t} + 2000 = 1332.07$$

$$e^{-0.05t} = 0.667930$$

$$t = -\frac{\ln 0.667930}{0.05} = 8.071437$$

Note that the present value of payments increases during the first 10 years. You see this from the second line above; $e^{-0.05t}$ has a negative coefficient and is a decreasing function of t , so the left side of the equation increases as t increases. Thus the present value of payments is greater than 1332.07 in the ranges $(8.071437, 10]$ and $(21.93438, \infty)$. The probability that death occurs in one of those ranges is $((10 - 8.071437) + (40 - 21.93438)) / 40 =$ $\boxed{0.49986}$.

- (c) For death right after time 10, the present value of the payments is

$$100\bar{a}_{\overline{10}|} = 100\left(\frac{1 - e^{-0.5}}{0.05}\right) = 786.94$$

For death at time $t \leq 10$, the present value of the payments is $2000 - 1000e^{-0.05t}$, which is always greater than 786.94. Therefore, $\boxed{786.94}$ is the minimum loss.

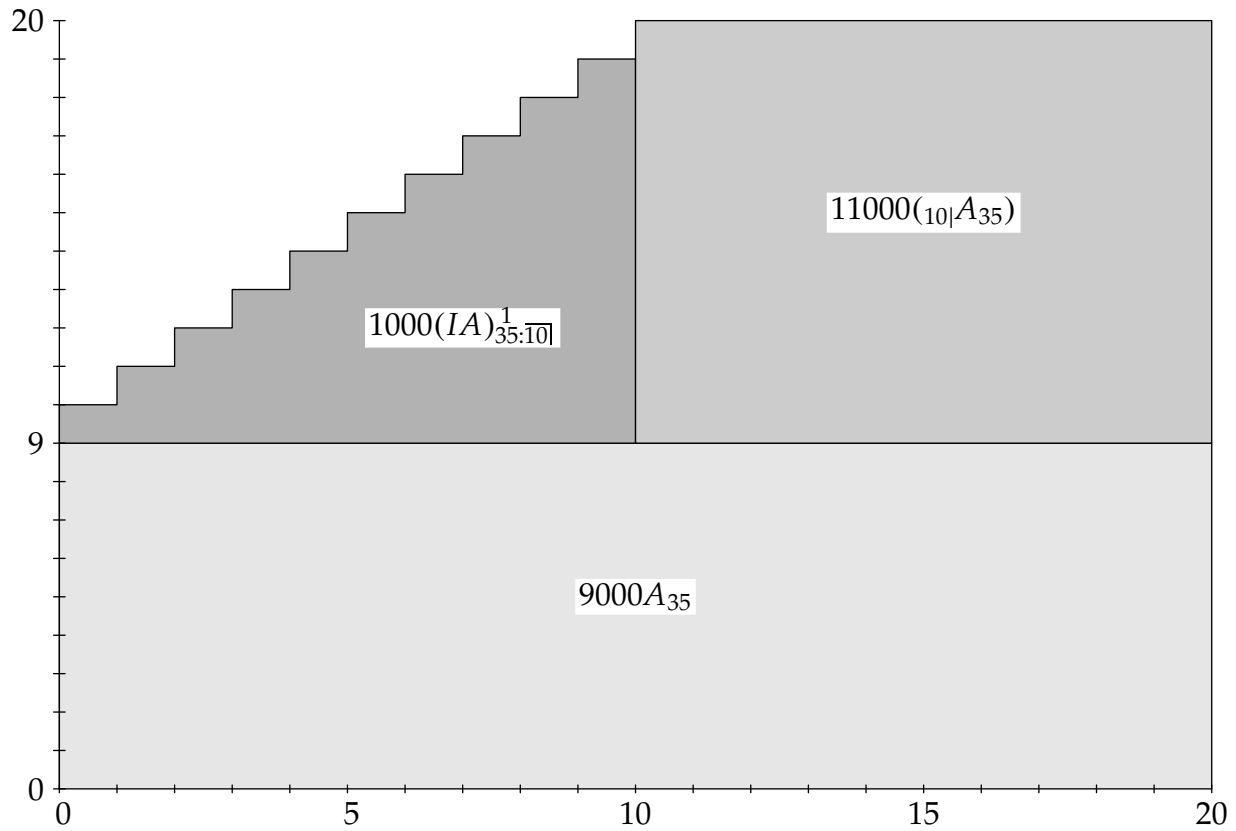


Figure A.1: Decomposition of increasing insurance in question 4

4. [Section 24.2 and Lesson 28]

- (a) The insurance can be expressed as a level whole life insurance of 9000, plus a 10-year increasing term insurance of 1000, plus a 10-year deferred insurance of 11,000. See figure A.1. Let A be the net single premium for the insurance payable at the end of the year of death.

$$\begin{aligned} A &= 9000A_{35} + 1000(IA)_{35:\overline{10}|}^1 + 11,000{}_{10}E_{35}A_{45} \\ &= 9000(0.09653) + 22.28 + 11,000(0.61069)(0.15161) = 1909.50 \end{aligned}$$

Multiplying by i/δ , we get $1.0248(1909.50) = \boxed{1956.85}$.

- (b)

$$\begin{aligned} \mu_{35} &\approx -0.5 \ln(l_{36}/l_{34}) = -0.5 \ln(99,517.80/99,593.80) = 0.0003817 \\ 12\ddot{a}_{35}^{(12)} &= 12 \left(18.9728 - \frac{11}{24} - \frac{143}{1728} (0.0003817 + \ln 1.05) \right) = \boxed{222.12} \end{aligned}$$

- (c) $1956.85/222.12 = \boxed{8.8097}$.

- (d) We need to calculate $20,000\bar{A}_{45}$ and $\ddot{a}_{45}^{(12)}$.

$$\begin{aligned} 20,000\bar{A}_{45} &= 1.0248(20,000)(0.15161) = 3107.39 \\ \mu_{45} &\approx -0.5 \ln(l_{46}/l_{44}) = -0.5 \ln(98,957.60/99,104.30) = 0.0007407 \\ 12\ddot{a}_{45}^{(12)} &= 12 \left(17.8162 - \frac{11}{24} - \frac{143}{1728} (0.0007407 + \ln 1.05) \right) = 208.25 \\ {}_{10}V &= 3107.39 - 8.8097(208.25) = \boxed{1272.81} \end{aligned}$$

- (e) This insurance can be decomposed into a 10-year decreasing insurance plus a 10-year deferred whole life insurance. The EPV of the decreasing insurance can be derived from

$$(IA)_{35:\overline{10}|} + (DA)_{35:\overline{10}|} = 11A_{35:\overline{10}|}$$

Let A be the net single premium for the insurance payable at the end of the year of death.

$$\begin{aligned} A &= 1000(11A_{35:\overline{10}|} - (IA)_{35:\overline{10}|}) + 1000_{10}E_{35}A_{45} \\ &= 1000(11(0.61464 - 0.61069) - 0.02228) + 1000(0.61069)(0.15161) \\ &= 95.2510 \end{aligned}$$

Multiplying by i/δ , we get $1.0248(95.2510) = \boxed{97.61}$.

An equivalent alternative is to evaluate the insurance as a whole life insurance for 11,000 minus a 10-year term increasing insurance for 1000 minus a 10-year deferred whole life insurance for 10,000.

5. (a) [Section 69.1] There are 380 lives with a full year of exposure at age 40. For the 6 lives in the table, exposure in months is (ages are written as yy:mm):

Birth date	Policy issue date	Withdrawal date	Death date	Exposure start	Exposure end	Exposure
Apr. 1, 1974	Feb. 1, 2015	—	—	40:10	40:12	2
June 1, 1974	Feb. 1, 2014	—	Feb. 1, 2015	40:7	40:12	5
Sept. 1, 1974	June 1, 2014	Aug. 1, 2015	—	40:4	40:11	7
Jan. 1, 1975	Jan. 1, 2008	—	May 1, 2015	40:0	40:12	12
Mar. 1, 1975	Mar. 1, 2011	Dec. 1, 2016	—	40:0	40:12	12
May 1, 1975	Dec. 1, 2005	Oct. 1, 2015	—	40:0	40:5	5

Notice that for those who die, exposure continues until the end of the year of age, or 40:12.

Actuarial exposure is $380 + \frac{2+5+7+12+12+5}{12} = 383\frac{7}{12}$ and $\hat{q}_{40} = \frac{2}{383\frac{7}{12}} = \boxed{0.005214}$.

- (b) [Lesson 66] We can read the entry and censoring times from the table in the solution to part (a). The deaths occur at times 40:8 and 40:4.

y_i	r_i	s_i
40:4	383	1
40:8	383	1

The Kaplan Meier estimate is

$$\hat{q}_{40} = 1 - \left(1 - \frac{1}{383}\right) \left(1 - \frac{1}{383}\right) = \boxed{0.005115}$$

- (c) [Section 68.1] Using Greenwood's formula,

$$(1 - 0.005115) \sqrt{\frac{1}{(383)(382)} + \frac{1}{(383)(382)}} = \boxed{0.003678}$$

- (d) [Lesson 69]
- No parametric distribution provides an adequate model.
 - Values of the survival function are only required at integers.
 - There is a large volume of data, almost all truncated or censored.

6. [Lesson 71]

- (a) For Cramer,
- ${}_{10}E_{55} = 0.59342$
- .

$$20(0.015(100,000) + 0.02(20,000))(0.59342)(11) = 248,049.6$$

For Liu, ${}_{30}E_{35} = (0.61069)(0.35994) = 0.21981$.

$$5((0.015)(50,000))(0.21981)(11) = 9,067.2$$

For Smith, ${}_{15}E_{50} = (0.77772)(0.59342) = 0.46151$.

$$10(0.015)(100,000)(0.46151)(11) = 76,149.9$$

Total actuarial liability is $248,049.6 + 9,067.2 + 76,149.9 = \boxed{333,267}$.

- (b) For Cramer, salary will be
- $120,000(1.03) = 123,600$
- next year. The discounted value of next year's liability is

$$21(0.015(100,000) + 0.02(23,600))(0.59342)(11) = 270,321.8$$

For Liu, salary will not exceed 100,000, so we can calculate the normal contribution directly using formula (71.2):

$$9,067.2 \left(1.03 \left(\frac{6}{5} \right) - 1 \right) = 2,139.9$$

For Smith, salary will be $100,000(1.03) = 103,000$ next year. The discounted value of next year's liability is

$$11(0.015(100,000) + 0.02(3,000))(0.59342)(11) = 87,115.5$$

The normal contribution is $(270,321.8 - 248,049.6) + 2,139.9 + (87,115.5 - 76,149.9) = \boxed{33,238}$.

- (c) Final salary is
- $120,000(1.03^{10}) = 161,270$
- . Annual pension is

$$30(0.015(100,000) + 0.02(61,270)) = 81,762$$

The replacement ratio is $81,762/161,270 = \boxed{0.5070}$.

- (d) Final salary is
- $100,000(1.03^{15}) = 155,797$
- . The annual payment under a monthly annuity-due is

$$25(0.015(100,000) + 0.02(55,797)) = 65,398$$

By Woolhouse's formula to two terms, $\ddot{a}_{65}^{(12)} = \ddot{a}_{65} - \frac{11}{24}$, so $\ddot{a}_{65} = 11\frac{11}{24}$, and

$$63,130\ddot{a}_{65}^{(12)} = x\ddot{a}_{65}$$

$$x = 65,398 \left(\frac{11}{11\frac{11}{24}} \right) = \boxed{62,782}$$

- (e) This change only affects Liu. We must recalculate
- ${}_{30}E_{35}$
- for Liu. We'll calculate it from first principles, although you may also calculate
- ${}_5E_{35}$
- and then multiply by
- ${}_{25}E_{40}$
- which can be calculated from the pure endowment columns of the Standard Ultimate Life Table.

$$\begin{aligned} {}_{30}p_{35} &= {}_4p_{35} {}_{p_{39}} {}_{25}p_{40} \\ &= \left(\frac{99,387.30}{99,556.70} \right) (1 - 0.00035) \left(\frac{94,579.70}{99,338.30} \right) = 0.950144 \\ {}_{30}E_{35} &= \frac{0.950144}{1.05^{30}} = 0.21984 \end{aligned}$$

The revised liability for Liu is

$$5((0.015)(50,000))(0.21984)(11) = 9068.5$$

instead of the previous 9067.2 calculated in part (a). The actuarial liability increases by 1 and becomes $\boxed{333,268}$.